Potential Game and Its Application to Control

Daizhan Cheng

Institute of Systems Science
Academy of Mathematics and Systems Science
Chinese Academy of Sciences

Seminar for SJTU Combinatorics Week
Shanghai Jiao Tong University
Shanghai, April 27, 2015

Outline of Presentation

I. An Introduction to Game Theory

☞ Game Theory

Figure 1: John von Neumann

J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944.

☞ Non-Cooperative Game
(Winner of Nobel Prize in Economics 1994)

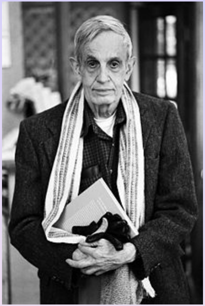


Figure 2: John Forbes Nash Jr.

J. Nash, Non-cooperative game, *The Annals of Mathematics*, Vol. 54, No. 2, 286-295, 1951.

☞ Cooperative Game
(Winner of Nobel Prize in Economics 2012 with Roth)

Figure 3: Lloyd S. Shapley

D. Gale, L.S. Shapley, Colle admissions and the stability of marriage, Vol. 69, American Math. Monthly, 9-15, 1962.

☞ Market Power and Regulation

(Winner of Nobel Prize in Economics 2014)

Figure 4: Jean Tirole

- D. Fudenberg and J. Tirole, Game Theory, MIT Press, Cambridge, MA, 1991.
- J. Tirole, The Theory of Industrial Organization, MIT Press, Cambridge, MA, 1988.

☞ Normal Non-cooperative Game

Definition 1.1

A **normal game** $G = (N, \mathcal{S}, c)$:

(i) **Player**: $N = \{1, 2, \cdots, n\}$.

(ii) **Strategy**: $$\mathcal{S}_i = \mathcal{D}_{k_i}, \quad i = 1, \cdots, n,$$

where

$$\mathcal{D}_k := \{1, 2, \cdots, k\}.$$

(iii) **Profile**: $$\mathcal{S} = \prod_{i=1}^{n} \mathcal{S}_i.$$

(iv) **Payoff function**:

$$c_j : \ \mathcal{S} \to \mathbb{R}, \quad j = 1, \cdots, n. \tag{1}$$
$$c := \{c_1, \cdots, c_n\}.$$

☞ Nash Equilibrium

Definition 1.2

In a normal game G, a profile

$$s = (x_1^*, \cdots, x_n^*) \in \mathcal{S}$$

is a **Nash equilibrium** if

$$c_j(x_1^*, \cdots, , x_j^*, \cdots, x_n^*) \geq c_j(x_1^*, \cdots, x_j, \cdots, x_n^*) \qquad j = 1, \cdots, n. \tag{2}$$

☞ Nash Equilibrium

Example 1.3

Consider a game G with two players: P_1 and P_2:

- Strategies of P_1: $\mathcal{D}_2 = \{1, 2\}$;
- Strategies of P_2: $\mathcal{D}_3 = \{1, 2, 3\}$.

Table 1: Payoff bi-matrix

$P_1 \backslash P_2$	1	2	3
1	2, 1	3, 2	6, 1
2	1, 6	2, 3	5, 5

$(1, 2)$ is a Nash equilibrium.

☞ Mixed Strategies

Definition 1.4

Assume the set of strategies for Player i is

$$S_i = \{1, \cdots, k_i\}.$$

Then Player i may take $j \in S_i$ with probability $r_j \geq 0$, $j = 1, \cdots, k_i$, where

$$\sum_{j=1}^{k_i} r_j = 1.$$

Such a strategy is called a **mixed strategy**. Denote by

$$x_i = (r_1, r_2, \cdots, r_{k_i})^T \in \Delta(S_i).$$

Notations

- **Mixed Strategy:**

$$\Upsilon_k := \left\{ (r_1, r_2, \cdots, r_k)^T \mid r_i \geq 0, \ \sum_{i=1}^{k} r_i = 1 \right\}.$$

- **Probabilistic Matrix:**

$$\Upsilon_{m\times n} := \left\{ M \in \mathcal{M}_{m\times n} \;\middle|\; \mathrm{Col}(M) \subset \Upsilon_m \right\}.$$

-

$$\mathbf{1}_m := (\underbrace{1, \cdots, 1}_{m})^T.$$

☞ Existence of Nash Equilibrium

Definition 1.5 (Nash 1950)

In the n-player normal game, $G = (N, S, c)$, if $|N|$ and $|S_i|$, $i = 1, \cdots, n$ are finite, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

II. Semi-tensor Product of Matrices

$$A_{m\times n} \times B_{p\times q} =?$$

Definition 2.1

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \operatorname{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \qquad (3)$$

☞ Important Comments

1. When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
2. STP keeps almost all the major properties of the conventional matrix product available.
    - Associativity, Distributivity;
    - $(A \ltimes B)^T = B^T \ltimes A^T$;
    - $(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}; \cdots$.

☞ Logical Variable and Logical Matrix

- *Vector Form of Logical Variables*: $x \in \mathcal{D}_k = \{1, 2, \cdots, k\}$, we identify

$$i \sim \delta_k^i, \quad i = 1, \cdots, k,$$

where δ_k^i is the i th column of I_k. Then $x \in \Delta_k$, where $\Delta_k = \{\delta_k^1, \cdots, \delta_k^k\}$.

- *Logical Matrix*:

$$L = [\delta_m^{k_1}, \delta_m^{k_2}, \cdots, \delta_m^{k_n}],$$

shorthand form:

$$L = \delta_m[k_1, k_2, \cdots, k_n].$$

☞ Matrix Expression of Logical Functions

Theorem 2.1

Let $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$ be a set of logical variables.

- Let $f : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ and

$$y = f(x_1, \cdots, x_n). \tag{4}$$

  Then there exists a unique matrix $M_f \in \mathcal{L}_{k_0 \times k}$ $(k = \prod_{i=1}^{n} k_i)$ such that in vector form

$$y = M_f \ltimes_{i=1}^{n} x_i := M_f x, \tag{5}$$

  where $x = \ltimes_{i=1}^{n} x_i$. M_f is called the structure matrix of f, and (5) is the algebraic form of (4).

☞ Matrix Expression of Pseudo-logical Functions

Theorem 2.1(cont'd)

- Let $c : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathbb{R}$ and

$$h = c(x_1, \cdots, x_n). \tag{6}$$

Then there exists a unique (row) vector $V_c \in \mathbb{R}^k$, such that in vector form

$$h = V_c x, \tag{7}$$

V_c is called the structure vector of c, and (7) is the algebraic form of (6)

☞ Khatri-Rao Product

Definition 2.2

Let $A \in \mathcal{M}_{p\times m}$, $B \in \mathcal{M}_{q\times m}$. Then the Khatri-Rao product of A and B is defined as

$$M * N := \left[\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \cdots \mathrm{Col}_m(M) \ltimes \mathrm{Col}_m(N)\right]. \quad (8)$$

☞ Matrix Expression of Logical Mapping

Let $x_i, y_j \in \mathcal{D}_k$, $i = 1, \cdots, n$, $j = 1, \cdots, m$, and $F : \mathcal{D}_k^n \to \mathcal{D}_k^m$ be

$$y_j = f_j(x_1, \cdots, x_n), \quad j = 1, \cdots, m. \tag{9}$$

Then in vector form we have

$$y_j = M_j x, \quad j = 1, \cdots, m. \tag{10}$$

Theorem 2.3

F can be expressed as

$$y = M_F x. \tag{11}$$

where $y = \ltimes_{j=1}^m y_j$, and

$$M_F = M_1 * M_2 * \cdots * M_m \in \mathcal{L}_{2^m \times 2^n}. \tag{12}$$

III. Potential Games

☞ Vector Space Structure of Finite Games

- $\mathcal{G}_{[n;k_1,\cdots,k_n]}$: the set of finite games with $|N|=n$, $|S_i|=k_i$, $i=1,\cdots,n$;
- In vector form: $x_i \in S_i = \Delta_{k_i}$, $i=1,\cdots,n$;
- $c_i : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathbb{R}$ can be expressed (in vector form) as

$$c_i(x_1,\cdots,x_n) = V_i^c \ltimes_{j=1}^n x_j, \quad i=1,\cdots,n,$$

  where V_i^c is the structure vector of c_i.
- Set

$$V_G := [V_1^c, V_2^c, \cdots, V_n^c] \in \mathbb{R}^{nk}.$$

  Then each $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is uniquely determined by V_G. Hence, $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ has a natural vector structure as

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} \sim \mathbb{R}^{nk}.$$

☞ Potential Games

Definition 3.1

Consider a finite game $G = (N, S, C)$. G is a potential game if there exists a function $P : S \to \mathbb{R}$, called the potential function, such that for every $i \in N$ and for every $s^{-i} \in S^{-i}$ and $\forall x, y \in S_i$

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = P(x, s^{-i}) - P(y, s^{-i}), \quad i = 1, \cdots, n. \tag{13}$$

D. Monderer, L.S. Shapley, Potential Games, *Games and Economic Behavior*, Vol. 14, 124-143, 1996.

☞ Fundamental Properties

Theorem 3.2

If G is a potential game, then the potential function P is unique up to a constant number. Precisely if P_1 and P_2 are two potential functions, then $P_1 - P_2 = c_0 \in \mathbb{R}$.

Theorem 3.3

Every finite potential game possesses a pure Nash equilibrium. Certain evolutions (Sequential or cascading MBRA) lead to a Nash equilibrium.

D. Monderer, L.S. Shapley, Potential games, Games Econ. Theory, 97, 81-108, 1996.

☞ Is a Game Potential?

Numerical computation ($n = 2$):

- Shapley (96): $O(k^4)$;
- Hofbauer (02): $O(k^3)$;
- Hilo (11): $O(k^2)$;
- Cheng (14): Potential Equation.

Hilo: "It is not easy, however, to verify whether a given game is a potential game."

- D. Monderer, L.S. Shapley, Potential games, Games Econ. Theory, 97, 81-108, 1996.
- J. Hofbauer, G. Sorger, A differential game approach to evolutionary equilibrium selection, Int. Game Theory Rev. 4, 17-31, 2002.
- Y. Hino, An improved algorithm for detecting potential games, Int. J. Game Theory, 40, 199-205, 2011.
- D. Cheng, On finite potential games, *Automatica*, Vol. 50, No. 7, 1793-1801, 2014.

Lemma 3.4

G is a potential game if and only if there exist $d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n)$, which is independent of x_i, such that

$$\begin{aligned} c_i(x_1, \cdots, x_n) = & P(x_1, \cdots, x_n) \\ & + d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n), \quad i = 1, \cdots, n, \end{aligned} \tag{14}$$

where P is the potential function.

Structure Vector Express:

$$\begin{array}{lcl} c_i(x_1, \cdots, x_n) & := & V_i^c \ltimes_{j=1}^n x_j \\ d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) & := & V_i^d \ltimes_{j \neq i} x_j, \quad i = 1, \cdots, n, \\ P(x_1, \cdots, x_n) & := & V_P \ltimes_{j=1}^n x_j. \end{array}$$

Define:

$$k^{[p,q]} := \begin{cases} \prod_{j=p}^{q} k_j, & q \geq p \\ 1, & q < p. \end{cases}$$

Construct:

$$\begin{array}{rcl} E_i & := & I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \\ & & \in \mathcal{M}_{k \times k/k_i}, \ i = 1, \cdots, n. \end{array} \tag{15}$$

Note that $\mathbf{1}_k \in \mathbb{R}^k$ is a column vector with all entries equal 1; $I_s \in \mathcal{M}_{s\times s}$ is the identity matrix and $I_1 := 1$.

$$\xi_i := \left(V_i^d\right)^T \in \mathbb{R}^{k^{n-1}}, \quad i = 1, \cdots, n. \tag{16}$$

☞ Potential Equation

Then (14) can be expressed as a linear system:

$$E\xi = b, \tag{17}$$

where

$$E = \begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}; \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}; \quad b = \begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix}. \tag{18}$$

(17) is called the potential equation and Ψ is called the potential matrix.

☞ Main Result

Theorem 3.5

A finite game G is potential if and only if the potential equation has solution. Moreover, the potential P can be calculated by

$$V_P = V_1^c - V_1^d (E_1)^T = V_1^c - \xi_1^T \left(\mathbf{1}_k^T \otimes I_k \right). \tag{19}$$

Example 3.6

Consider a prisoner's dilemma with the payoff bi-matrix as in Table 2.

Table 2: Payoff Bi-matrix of Prisoner's Dilemma

$P_1 \backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

Example 3.6 (cont'd)

From Table 2

$$V_1^c = (R, S, T, P)$$
$$V_2^c = (R, T, S, P).$$

Assume $V_1^d = (a, b)$ and $V_2^d = (c, d)$. It is easy to calculate that

$$\begin{aligned} E_1 &= \delta_2[1, 2, 1, 2]^T, \\ E_2 &= \delta_2[1, 1, 2, 2]^T. \end{aligned}$$

$$b_2 = (V_2^c - V_1^c)^T = (0, T - S, S - T, 0)^T.$$

Example 3.6 (cont'd)

Then the potential equation (18) becomes

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ T-S \\ S-T \\ 0 \end{bmatrix}. \tag{20}$$

Example 3.6 (cont'd)

It is easy to solve it out as

$$\begin{cases} a = c = T - c_0 \\ b = d = S - c_0 \end{cases}$$

where $c_0 \in \mathbb{R}$ is an arbitrary number. We conclude that the general **Prisoner's Dilemma is a potential game**.

Using (19), the potential can be obtained as

$$\begin{aligned} V_P &= V_1^c - V_1^d D_f^{[2,2]} \\ &= (R - T, 0, 0, P - S) + c_0(1, 1, 1, 1). \end{aligned} \tag{21}$$

From (17), G is potential if and only if

$$\begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \in \mathrm{Span}(E). \tag{22}$$

Since V_1^c is free, we have

$$\begin{bmatrix} (V_1^c)^T \\ (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \in \mathrm{Span}(E^e), \tag{23}$$

where

$$E^e = \begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}.$$

Equivalently, we have

$$\begin{bmatrix} I_k & 0 & \cdots & 0 \\ -I_k & I_k & \cdots & 0 \\ \vdots & & \ddots & \\ -I_k & 0 & \cdots & I_k \end{bmatrix} \begin{bmatrix} (V_1^c)^T \\ (V_2^c)^T \\ (V_3^c)^T \\ \vdots \\ (V_n^c)^T \end{bmatrix} \in \mathrm{Span}(E^e). \tag{24}$$

That is

$$V_G^T \in \mathrm{Span}(E_P), \tag{25}$$

where

$$
\begin{aligned}
E_P &:= \begin{bmatrix} I_k & 0 & \cdots & 0 \\ -I_k & I_k & \cdots & 0 \\ \vdots & & \ddots & \\ -I_k & 0 & \cdots & I_k \end{bmatrix}^{-1} E^e \\
&= \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix} .
\end{aligned} \tag{26}
$$

E_n^0 is obtained from E_n by deleting the last column, and define

$$E_P^0 := \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n^0 \end{bmatrix}.$$

Then we have

$$\mathrm{Span}(E_P) = \mathrm{Span}(E_P^0).$$

Moreover, it is easy to see that the columns of E_P^0 are linearly independent.

☞ Potential Subspace

Theorem 3.7

The subspace of potential games is

$$\mathcal{G}_P = \mathrm{Span}(E_P), \tag{27}$$

which has $\mathrm{Col}(E_P^0)$ as its basis.

According to the construction of E_P^0 it is clear that

Corollary 3.8

The dimension of the subspace of potential games of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ is

$$\dim(\mathcal{G}_P) = k + \sum_{j=1}^{n} \frac{k}{k_j} - 1. \tag{28}$$

IV. Decomposition of Finite Games

☞ Non-strategic Games

Definition 4.1

Let $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. G and $\tilde{G}$ are said to be strategically equivalent, if for any $i \in N$, any $x_i,\ y_i \in S_i$, and any $x^{-i} \in S^{-i}$, (where $S^{-i} = \prod_{j \neq i} S_j$), we have

$$c_i(x_i, x^{-i}) - c_i(y_i, x^{-i}) = \tilde{c}_i(x_i, x^{-i}) - \tilde{c}_i(y_i, x^{-i}), \quad i = 1, \cdots, n. \tag{29}$$

Lemma 4.2

Two games $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ are strategically equivalent, if and only if for each $x^{-i} \in S^{-i}$ there exists $d_i(x^{-i})$ such that

$$\begin{array}{l} c_i(x_i, x^{-i}) - \tilde{c}_i(x_i, x^{-i}) = d_i(x^{-i}), \\ \forall x_i \in S_i,\ \forall x^{-i} \in S^{-i},\ i = 1, \cdots, n. \end{array} \tag{30}$$

Theorem 4.3

G and $\tilde{G}$ are strategically equivalent if and only if

$$\left(V_G^c - V_{\tilde{G}}^c\right)^T \in \mathrm{Span}\left(B_N\right), \tag{31}$$

where

$$B_N = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{32}$$

Definition 4.4

The subspace

$$\mathcal{N} := \mathrm{Span}(B_N)$$

is called the non-strategic subspace.

Corollary 4.5

The dimension of $\mathcal{N}$ is

$$\dim(\mathcal{N}) = \sum_{i=1}^{n} \frac{k}{k_i}. \tag{33}$$

Define

$$\tilde{E}_P := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{34}$$

Comparing (34) with (26), it is ready to verify that

$$\mathcal{G}_P = \mathrm{Span}\left(\tilde{E}_P\right) = \mathrm{Span}\left(E_P\right). \tag{35}$$

Deleting the last column of $\tilde{E}_P$, (equivalently, replacing the E_n in $\tilde{E}_P$ by E_n^0), the remaining matrix is denoted as

$$\tilde{E}_P^0 := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n^0 \end{bmatrix}. \tag{36}$$

Then it is clear that $\mathrm{Col}\left(\tilde{E}_P^0\right)$ is a basis of $\mathcal{G}_P$.

Observing (34) again, it follows immediately that

Corollary 4.6

The subspace $\mathcal{N}$ is a linear subspace of $\mathcal{G}_P$. That is,

$$\mathcal{N} \subset \mathcal{G}_P.$$

☞ Orthogonal Decomposition

Theorem 4.7

(Candogan et al, 2011)

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{Potential \quad games} \overbrace{\quad \oplus \quad \mathcal{H}}^{Harmonic \quad games}. \tag{37}$$

O. Candogan, I. Menache, A. Ozdaglar, P.A. Parrilo, Flows and decompositions of games: Harmonic and potential games, *Mathematcs of Operations Research*, Vol. 36, No. 3, 474-503, 2011.

☞ Pure Potential Games $\mathcal{P}$

Using (34)-(35), we have

$$\begin{aligned}\mathcal{G}_P &= \text{Span}(\tilde{E}_P)\\ &= \text{Span}\begin{bmatrix} I_k - \frac{1}{k_1}E_1E_1^T & E_1 & 0 & 0 & \cdots & 0\\ I_k - \frac{1}{k_2}E_2E_2^T & 0 & E_2 & 0 & \cdots & 0\\ I_k - \frac{1}{k_3}E_3E_3^T & 0 & 0 & E_3 & \cdots & 0\\ \vdots & & & & \ddots & \\ I_k - \frac{1}{k_n}E_nE_n^T & 0 & 0 & 0 & \cdots & E_n \end{bmatrix}.\end{aligned} \tag{38}$$

$$B_P = \begin{bmatrix} I_k - \frac{1}{k_1} E_1 E_1^T \\ I_k - \frac{1}{k_2} E_2 E_2^T \\ \vdots \\ I_k - \frac{1}{k_n} E_n E_n^T \end{bmatrix} \in \mathcal{M}_{nk \times k}. \tag{39}$$

Then we have

$$\mathcal{P} = \mathcal{V} = \mathrm{Span}\left(B_P\right). \tag{40}$$

Since $\dim(\mathcal{P}) = k-1$, to find the basis of $\mathcal{P}$ one column of V needs to be removed. Note that

$$\begin{aligned}
& \left(I_k - \frac{1}{k_i} E_i E_i^T\right) \mathbf{1}_k \\
= & \ \left(I_{k[1,i-1]} \mathbf{1}_{k[1,i-1]}\right) \left[\left(I_{k_i} - \frac{1}{k_i} \mathbf{1}_{k_i \times k_i}\right) \mathbf{1}_{k_i}\right] \\
& \left(I_{k[i+1,n]} \mathbf{1}_{k[i+1,n]}\right) \\
= & \ 0, \quad i = 1, \cdots, n.
\end{aligned}$$

It follows that

$$B_P \mathbf{1}_{nk} = 0.$$

Deleting any one column of B_P, say, the last column, and denoting the remaining matrix by B_P^0, then we know that

Theorem 4.8

$$\mathcal{P} = \mathrm{Span}\left(B_P\right) = \mathrm{Span}\left(B_P^0\right),$$

where B_P^0 is a basis of $\mathcal{P}$.

☞ Pure Harmonic Games $\mathcal{H}$

we can construct a set of vectors, which are in $\mathcal{G}_P^{\perp}$ as

$$J_1 := \left\{ \begin{array}{c} \begin{bmatrix} (\delta_{k_1}^{1} - \delta_{k_1}^{i_1})(\delta_{k_2}^{1} - \delta_{k_2}^{i_2})\delta_{k_3}^{i_3} \cdots \delta_{k_n}^{i_n} \\ -(\delta_{k_1}^{1} - \delta_{k_1}^{i_1})(\delta_{k_2}^{1} - \delta_{k_2}^{i_2})\delta_{k_3}^{i_3} \cdots \delta_{k_n}^{i_n} \\ \mathbf{0}_{(n-2)k} \end{bmatrix} \\ i_1 \neq 1, i_2 \neq 1 \end{array} \right\};$$

$$J_2 := \left\{ \begin{array}{c} \begin{bmatrix} (\delta^1_{k_1} - \delta^{i_1}_{k_1})\delta^1_{k_2}(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^{i_4}_{k_4}\cdots\delta^{i_n}_{k_n} \\ \delta^{i_1}_{k_1}(\delta^1_{k_2} - \delta^{i_2}_{k_2})(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^{i_4}_{k_4}\cdots\delta^{i_n}_{k_n} \\ -(\delta^1_{k_1}\delta^1_{k_2} - \delta^{i_1}_{k_1}\delta^{i_2}_{k_2})(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^{i_4}_{k_4}\cdots\delta^{i_n}_{k_n} \\ \mathbf{0}_{(n-3)k} \end{bmatrix} \\ (i_1, i_2) \neq \mathbf{1}_2^T; i_3 \neq 1 \end{array} \right\};$$

$$\vdots$$

$$
J_{n-1} :=
\left\{ \begin{bmatrix}
(\delta^1_{k_1} - \delta^{i_1}_{k_1})\delta^1_{k_2}\delta^1_{k_3}\delta^1_{k_4}\cdots\delta^1_{k_{n-1}}(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
\delta^{i_1}_{k_1}(\delta^1_{k_2} - \delta^{i_2}_{k_2})\delta^1_{k_3}\delta^1_{k_4}\cdots\delta^1_{k_{n-1}}(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^1_{k_4}\cdots\delta^1_{k_{n-1}}(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
\vdots \\
\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\delta^{i_3}_{k_3}\delta^{i_4}_{k_4}\cdots(\delta^1_{k_{n-1}} - \delta^{i_{n-1}}_{k_{n-1}})(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
-(\delta^1_{k_1}\delta^1_{k_2}\cdots\delta^1_{k_{n-1}} - \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\cdots\delta^{i_{n-1}}_{k_{n-1}})(\delta^1_{k_n} - \delta^{i_n}_{k_n})
\end{bmatrix} \\
(i_1, \cdots, i_{n-1}) \neq \mathbf{1}^T_{n-1}; i_n \neq 1 \right\}.
$$

Define

$$B_H := [J_1, J_2, \cdots, J_{n-1}]. \tag{41}$$

Then we can show B_H is the basis of $\mathcal{H}$:

Theorem 4.9

B_H has full column rank and

$$\mathcal{H} = \mathrm{Span}(B_H). \tag{42}$$

Theorem 4.10

$G \in \mathcal{H}$, iff

- $$\sum_{i=1}^{n} c_i(s) = 0, \quad s \in S; \tag{43}$$

- $$\sum_{x \in S_i} c_i(x,y) = 0, \quad \forall y \in S^{-i};\ i = 1, \cdots, n. \tag{44}$$

☞ Nash Equilibrium of $\mathcal{G}_H$

Definition 4.11

Let $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ and $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ a Nash equilibrium of G. s^* is called a flat Nash equilibrium, if

$$\begin{aligned} c_i(s_1^*, s_2^*, \cdots, s_n^*) &= c_i(s_1^*, s_2^*, \cdots, s_i, \cdots, s_n^*), \\ & \forall s_i \in S_i;\ i = 1, \cdots, n. \end{aligned}$$

A flat Nash equilibrium is called a zero Nash equilibrium if

$$c_i(s_1^*, s_2^*, \cdots, s_n^*) = 0, \quad i = 1, \cdots, n.$$

Example 4.12

Consider $G \in \mathcal{G}_{[2;k_1,k_2]}$. Assume (s_1^*, s_2^*) is a flat Nash equilibrium, then the payoff bi-matrix is as Table 3:

Table 3: Flat Nash Equilibrium

$P_1 \backslash P_2$	1	2	$\cdots$	s_2^*	$\cdots$	k_2
1	$(\times, \times)$	$(\times, \times)$	$\cdots$	$(a, \times)$	$\cdots$	$(\times, \times)$
2	$(\times, \times)$	$(\times, \times)$	$\cdots$	$(a, \times)$	$\cdots$	$(\times, \times)$
$\vdots$			$\vdots$		$\vdots$	
s_1^*	$(\times, b)$	$(\times, b)$	$\cdots$	(a, b)	$\cdots$	$(\times, b)$
$\vdots$			$\vdots$		$\vdots$	
k_1	$(\times, \times)$	$(\times, \times)$	$\cdots$	$(a, \times)$	$\cdots$	$(\times, \times)$

As $a = b = 0$, (s_1^*, s_2^*) is a zero Nash equilibrium.

☞ Nash Equilibriums of $\mathcal{G}_H = \mathcal{H} \oplus \mathcal{N}$

Theorem 4.13

1. If $G \in \mathcal{N}$, then every strategy profile is a flat Nash equilibrium;
2. If $G \in \mathcal{H}$ and s^* is a Nash equilibrium, then s^* is a zero Nash equilibrium;
3. If $G \in \mathcal{G}_H$ and s^* is a Nash equilibrium, then s^* is a flat Nash equilibrium.

☞ Networked Evolutionary Game (NEG)

Definition 5.1

A **networked evolutionary game**, denoted by $((N,E),G,\Pi)$, consists of

(i) a network graph (N,E);

(ii) a fundamental network game (FNG), G, such that if $(i,j) \in E$, then i and j play FNG with strategies $x_i(t)$ and $x_j(t)$ respectively;

(iii) a local information based strategy updating rule (SUR).

☞ Network Graph: (N, E)

Definition 5.2

1. (N, E) is a graph, where N is the set of nodes and $E \subset N \times N$ is the set of edges.
2. $U_d(i) = \{j \,|\,$ there is a path connecting i, j with length $\leq d\}$
3. $U_0(i) := \{i\}; \quad U_1(i) = U(i); \quad U_\alpha(i) \subset U_\beta(i), \ \alpha \leq \beta.$
4. If $(i, j) \in E$ implies $(j, i) \in E$ the graph is undirected, otherwise, it is directed.

Definition 5.3

A network is **homogeneous**, if each node has the same degree (for undirected graph) / in-degree and out-degree (for directed graph).

☞ Fundamental Network Game: G

Definition 5.4

A normal game with two players is called a **fundamental network game** (FNG), if

$$S_1 = S_2 := S_0 = \{1, 2, \cdots, k\}.$$

☞ Overall Payoff

$$c_i(t) = \sum_{j \in U(i) \backslash i} c_{ij}(t), \quad i \in N. \tag{45}$$

☞ Strategy Updating Rule: Π

Definition 5.5

A **strategy updating rule** (SUR) for an NEG, denoted by Π, is a set of mappings:

$$x_i(t+1) = g_i\left(x_j(t), c_j(t) \,\middle|\, j \in U(i)\right), \quad t \geq 0, \quad i \in N. \tag{46}$$

Remark 5.6

1. g_i could be a probabilistic mapping (*i.e.*, a mixed strategy is used);
2. When the network is homogeneous, g_i, $i \in N$, are the same.

☞ Strategy Profile Dynamics

Since $c_j(t)$ depends on $x_\ell(t)$, $\ell \in U(j)$, (46) can be expressed as

$$x_i(t+1) = f_i\left(x_j(t) \,\middle|\, j \in U_2(i)\right), \quad t \geq 0, \quad i \in N. \tag{47}$$

Now (47) is a standard k-valued logical dynamic system, its profile dynamics can be expressed as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)). \end{cases} \tag{48}$$

D. Cheng, F. He, H. Qi, T. Xu. Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Aut. Contr.*, (in print), On line: DOI:10.1109/TAC.2015.2404471.

☞ Potential NEG

Theorem 5.7

Consider an NEG, $((N,E),G,\Pi)$. If the fundamental network game G is potential, then the NEG is also potential. Moreover, the potential P of the NEG is:

$$P(s) := \sum_{(i,j)\in E} P^{i,j}(s_i, s_j). \tag{49}$$

Example 5.8

Consider an NEG $((N, E), G, \Pi)$, where the network graph is described as in Fig. 5.

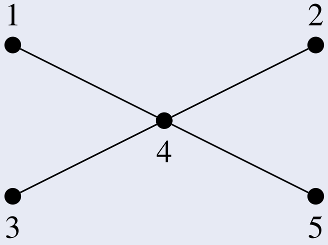


Figure 5: Network Graph

Example 5.8 (cont'd)

Assume:

- G: the prisoner's dilemma with $R = -1$, $S = -10$, $T = 0$, $P = -5$.
- Π: MBRA (Potential $\Rightarrow$ Pure Nash Equalibrium)

$$\Psi = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{128\times 80}.$$

Example 5.8 (cont'd)

It is easy to check that

$$V_1^c = \begin{bmatrix} -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & -5 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & 5 \end{bmatrix}.$$

$$V_2^c = \begin{bmatrix} -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & -5 \end{bmatrix}.$$

Example 5.8 (cont'd)

$$V_3^c = \begin{bmatrix} -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \end{bmatrix}.$$

$$V_4^c = \begin{bmatrix} -4 & -13 & 0 & -5 & -13 & -22 & -5 & -10 \\ -13 & -22 & -5 & -10 & -22 & -31 & -10 & -15 \\ -13 & -22 & -5 & -10 & -22 & -31 & -10 & -15 \\ -22 & -31 & -10 & -15 & -31 & -40 & -15 & -20 \end{bmatrix}.$$

Example 5.8 (cont'd)

$$V_5^c = [-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5$$
$$-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5$$
$$-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5$$
$$-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5].$$

It is easy check that the networked game is potential.

Example 5.8 (cont'd)

Moreover,

$$\xi_1 = \begin{bmatrix} 28 & 27 & 15 & 10 & 27 & 26 & 10 & 5 \\ 27 & 26 & 10 & 5 & 26 & 25 & 5 & 0 \end{bmatrix}.$$

Using potential formula, we have

$$V_P = \begin{bmatrix} -29 & -28 & -25 & -20 & -28 & -27 & -20 & -15 \\ -28 & -27 & -20 & -15 & -27 & -26 & -15 & -10 \\ -28 & -27 & -20 & -15 & -27 & -26 & -15 & -10 \\ -27 & -26 & -15 & -10 & -26 & -25 & -10 & -5 \end{bmatrix}.$$

Example 5.8 (cont'd)

Calculating P separately.
First, for any $(i,j) \in E$ we have

$$P(x_i, x_j) = V_0 x_i x_j, \tag{50}$$

where

$$V_0 = (R - T, 0, 0, P - S) = (-1\ 0\ 0\ 5).$$

Next, we have

$$\begin{aligned} V_P^{1,2} &= V_0 D_r^{[4,8]} = V_0 \left(I_4 \otimes \mathbf{1}_8^T\right) \\ &= [-1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \\ &\quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \\ &\quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \\ &\quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5]. \end{aligned}$$

Example 5.8 (cont'd)

Similarly, we can figure out all $V_P^{i,j}$ as

$$\begin{aligned}
V_P^{1,3} &= V_0 D_r^{[2,2]} D_r^{[8,2]}, & V_P^{1,4} &= V_0 D_r^{[2,4]} D_r^{[16,2]}, \\
V_P^{1,5} &= V_0 D_r^{[2,8]}, & V_P^{2,3} &= V_0 D_f^{[2,2]} D_r^{[8,4]}, \\
V_P^{2,4} &= V_0 D_f^{[2,2]} D_r^{[4,2]} D_r^{[16,2]}, & V_P^{2,5} &= V_0 D_f^{[2,2]} D_r^{[4,4]} \\
V_P^{3,4} &= V_0 D_f^{[4,2]} D_r^{[16,2]}, & V_P^{3,5} &= V_0 D_f^{[4,2]} D_r^{[8,2]}, \\
V_P^{4,5} &= V_0 D_f^{[8,2]}.
\end{aligned}$$

Example 5.8 (cont'd)

$$\begin{aligned} V_{\tilde{P}} &= V_P^{1,4} + V_P^{2,4} + V_P^{3,4} + V_P^{4,5} \\ &= [\begin{matrix} -4 & -3 & 0 & 5 & -3 & -2 & 5 & 10 \\ -3 & -2 & 5 & 10 & -2 & -1 & 10 & 15 \\ -3 & -2 & 5 & 10 & -2 & -1 & 10 & 15 \\ -2 & -1 & 10 & 15 & -1 & 0 & 15 & 20 \end{matrix}]. \end{aligned}$$

Comparing this result with the above V_P, one sees easily that

$$\tilde{P}(x) = P(x) + 25.$$

VI. Applications

☞ Consensus of MAS

- Network graph: $(N, E(t))$: $N = \{1, 2, \cdots, n\}$ with varying topology: $E(t)$.
- Model of MAS:

$$a_i(t+1) = f_i\left(a_j(t) | j \in U(i)\right), \quad i = 1, \cdots, n. \tag{51}$$

- Set of Strategies:

$$a_i \in \mathcal{A}_i \subset \mathbb{R}^n, \quad i = 1, \cdots, n.$$

J.R. Marden, G. Arslan, J. S. Shamma, Cooperative control and potential games, *IEEE Trans. Sys., Man, Cybernetcs, Part B*, Vol. 39, No. 6, 1393-1407, 2009.

☞ Distributed Coverage of Graphs

- Unknown connected graph $\mathcal{G} = (V, E)$.
- Mobile agents $N = \{1, 2, \cdots, n\}$ (initially arbitrarily deployed on $\mathcal{G}$).
- Agent a_i can cover $U^i(t) := U_{d_i}(a_i(t))$, $i = 1, \cdots, n$.

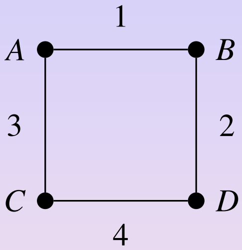
Purpose: $\max_a \bigcup_{i=1}^n U^i$.

A.Y. Yazicioglu, M. Egerstedt, J.S. Shamma, A game theoretic approach to distributed coverage of graphs by heterogeneous mobile agents, *Est. Contr. Netw. Sys.*, Vol. 4, 309-315, 2013.

M. Zhu, S. Martinez, Distributed coverage games for energy-aware mobile sensor networks, *SIAM J. Cont. Opt.*, Vol. 51, No. 1, 1-27, 2013.

☞ Congestion Games

Problem: Player 1 want to go from A to D, player 2 want to go from B to C:



- D. Monderer, L.S. Shapley, Potential Games, *Games & Economic Behavior*, Vol. 14, 124-143, 1996.
- X. Wang, N. Xiao, et al, Distributed consensus in noncooperative congestion games: an application to road pricing, *Proc. 10th IEEE Int. Conf. Contr. Aut.*, Hangzhou, China, 1668-1673, 2013.

V. Conclusion

- Formulas for verifying and calculating potential function are obtained.
- Vector space structure of finite non-cooperative games is introduced. Its decomposition is investigated.

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{Potential \quad games}\!\!\!\!\overbrace{\quad \oplus \quad \mathcal{H}}^{Harmonic \quad games}.$$

- The Nash equilibriums of $\mathcal{G}_H = \mathcal{H} \oplus \mathcal{N}$ are explored.
- The strategy profile dynamics of an NEG is derived. Properties of certain (potential) NEGs are studied.
- Three applications for potential NEGs are introduced.

Last Comments:

Game-based Control or Control Oriented Game could be a challenging new direction for Control Community.

Thank you for your attention!

Question?