STP Approach to Finite Game

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Outline of Presentation

1. An Introduction to Game Theory
2. Networked Evolutionary Game
3. STP and its Applications to Finite Games
4. Potential Game
5. Decomposition of Finite Games
6. Static Bayesian Game
7. Game Theoretic Control
8. Concluding Remarks

I. An Introduction to Game Theory

☞ Game Theory

图 **1:** John von Neumann

J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944.

☞ Non-Cooperative Game

(Winner of Nobel Prize in Economics 1994)

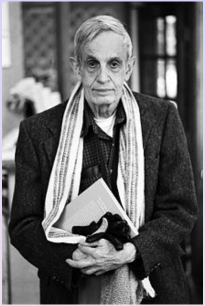

图 **2:** John Forbes Nash Jr.

J. Nash, Non-cooperative game, *The Annals of Mathematics*, Vol. 54, No. 2, 286-295, 1951.

☞ Finite (Non-Cooperative) Game

Definition 1.1

A finite game $G = (N, \mathcal{S}, c)$:

(i) Player: $N = \{1, 2, \cdots, n\}$.

(ii) Strategy:
$$\mathcal{S}_i = \mathcal{D}_{k_i}, \quad i = 1, \cdots, n,$$
where
$$\mathcal{D}_k := \{1, 2, \cdots, k\}.$$
Profile:
$$\mathcal{S} = \prod_{i=1}^{n} \mathcal{S}_i.$$

(iii) Payoff (utility) function:
$$c_j : \mathcal{S} \to \mathbb{R}, \quad j = 1, \cdots, n. \tag{1}$$
$$c := \{c_1, \cdots, c_n\}.$$

☞ Nash Equilibrium

Definition 1.2

In a normal game G, a profile

$$s = (x_1^*, \cdots, x_n^*) \in \mathcal{S}$$

is a Nash equilibrium if

$$c_j(x_1^*, \cdots, , x_j^*, \cdots, x_n^*) \geq c_j(x_1^*, \cdots, x_j, \cdots, x_n^*) \qquad j = 1, \cdots, n. \tag{2}$$

No player has anything to gain by unilateral change of strategy if the strategies of the others remain unchanged. Hence, they have no incentive to change.

☞ Nash Equilibrium

Example 1.3

Consider a game G with two players: P_1 and P_2:

- Strategies of P_1: $\mathcal{D}_2 = \{1, 2\}$;
- Strategies of P_2: $\mathcal{D}_3 = \{1, 2, 3\}$.

表 **1:** Payoff bi-matrix

$P_1 \backslash P_2$	1	2	3
1	2, 1	3, 2	6, 1
2	1, 6	2, 3	5, 5

$(1, 2)$ is a Nash equilibrium.

☞ Mixed Strategies

Definition 1.4

Assume the set of strategies for Player i is

$$S_i = \{1, \cdots, k_i\}.$$

Then Player i may take $j \in S_i$ with probability $r_j \geq 0$, $j = 1, \cdots, k_i$, where

$$\sum_{j=1}^{k_i} r_j = 1.$$

Such a strategy is called a mixed strategy. Denote by

$$x_i = (r_1, r_2, \cdots, r_{k_i})^T \in \overline{S}_i.$$

☞ Notations

- **Pure Strategy:**

$$\Delta_k; = \{\delta_k^i \mid i = 1, 2, \cdots, k\}$$

- **Logical Matrix:**

$$\mathcal{L}_{k\times n}; = \{[\delta_k^{i_1}, \delta_k^{i_2}, \cdots, \delta_k^{i_n}] = \delta_k[i_1, i_2, \cdots, i_n]\}$$

- **Mixed Strategy:**

$$\Upsilon_k := \{(r_1, r_2, \cdots, r_k)^T \mid r_i \geq 0, \ \sum_{i=1}^{k} r_i = 1\}.$$

- **Probabilistic Matrix:**

$$\Upsilon_{k\times n} := \left\{M \in \mathcal{M}_{k\times n} \,\middle|\, \mathrm{Col}(M) \subset \Upsilon_k\right\}.$$

☞ Existence of Nash Equilibrium

Theorem 1.5

(Nash 1950) In the n-player normal game, $G = (N, S, c)$, if $|N|$ and $|S_i|$, $i = 1, \cdots, n$ are finite, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

Example 1.6

Consider Rock-Paper-Scissors, where

- $N = \{P_1,\ P_2\}$;
- $S_i = \{R, S, P\}$, $i = 1, 2$;
- Payoff: payoffs are described by the following payoff bi-matrix.

Example 1.6(cont'd)

表 **2:** Payoff Bi-matrix of R-P-S

$P_1 \backslash P_2$	$R=1$	$S=2$	$C=3$
$R=1$	$(0,\ 0)$	$(1,\ -1)$	$(-1,\ 1)$
$S=2$	$(-1,\ 1)$	$(0,\ 0)$	$(1,\ -1)$
$C=3$	$(1,\ -1)$	$(-1,\ 1)$	$(0,\ 0)$

It is easy to check that

- there is no pure Nash equilibrium;
- There is a mixed Nash equilibrium

$$s_i = (1/3, 1/3, 1/3), \quad i = 1, 2.$$

II. Networked Evolutionary Game

☞ Evolutionary Game

Strategy Profile Dynamics (as $G = (N, S, c)$ is repeated):

$$\begin{array}{rcl} x_i(t+1) & = & f_i(x_j(s), c_j(s) | j = 1, \cdots, n;\ s = 0, 1, \cdots, t) \\ & & i = 1, \cdots, n, \end{array} \tag{3}$$

where $x_i \in S_i$, $i = 1, \cdots, n$.

Markovian Strategy Profile Dynamics:

$$\begin{array}{rcl} x_i(t+1) & = & f_i(x_j(t), c_j(t) | j = 1, \cdots, n) \\ & & i = 1, \cdots, n, \end{array} \tag{4}$$

where $x_i \in S_i$, $i = 1, \cdots, n$.

☞ Networked Evolutionary Game (NEG)

Definition 2.1

A networked evolutionary game, denoted by $((N,E),G,\Pi)$, consists of

(i) a network graph (N,E);

(ii) a fundamental network game (FNG), G, such that if $(i,j)\in E$, then i and j play FNG with strategies $x_i(t)$ and $x_j(t)$ respectively;

(iii) a local information based strategy updating rule (SUR).

D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Aut. Contr* Vol. 60, No. 9, 2402-2415, 2015.

☞ Network Graph: (N, E)

Definition 2.2

1. (N, E) is a graph, where N is the set of nodes and $E \subset N \times N$ is the set of edges.
2. $U_d(i) = \{j \mid \text{there is a path connecting } i, j \text{ with length } \leq d\}$
3. $U_0(i) := \{i\}; \quad U_1(i) = U(i); \quad U_\alpha(i) \subset U_\beta(i), \ \alpha \leq \beta.$
4. If $(i, j) \in E$ implies $(j, i) \in E$ the graph is undirected, otherwise, it is directed.

Definition 2.3

A network is homogeneous, if each node has the same degree (for undirected graph) / in-degree and out-degree (for directed graph).

☞ Fundamental Network Game: G

Definition 2.4

(i) A normal game with two players is called a fundamental network game (FNG), if

$$S_1 = S_2 := S_0 = \{1, 2, \cdots, k\}.$$

(ii) An FNG is symmetric, that is,

$$c_1(x, y) = c_2(y, x), \quad \forall x, y \in S_0.$$

☞ Overall Payoff

$$c_i(t) = \sum_{j \in U(i) \setminus i} c_{ij}(t), \quad i \in N. \tag{5}$$

☞ Strategy Updating Rule

:

Definition 2.5

A strategy updating rule (SUR) for an NEG, denoted by Π, is a set of mappings:

$$x_i(t+1) = g_i\left(x_j(t), c_j(t) \,\middle|\, j \in U(i)\right), \quad t \geq 0, \quad i \in N. \tag{6}$$

Remark 2.6

1. g_i could be a probabilistic mapping (*i.e.*, a mixed strategy is used);
2. When the network is homogeneous, g_i, $i \in N$, are the same.

☞ Some SURs

(1) $\Pi - I$: Unconditional Imitation with fixed priority (UI-1).

$$j^* = \text{argmax}_{j \in U(i)} c_j(x(t)), \tag{7}$$

$$\Rightarrow \quad x_i(t+1) = x_{j^*}(t). \tag{8}$$

Optimal payment neighbors are not unique:

$$\text{argmax}_{j \in U(i)} c_j(x(t)) := \{j_1^*, \cdots, j_r^*\},$$

set priority:

$$j^* = \min\{\mu | \mu \in \text{argmax}_{j \in U(i)} c_j(x(t))\}. \tag{9}$$

$\Rightarrow$ Deterministic k-valued dynamics.

☞ Some SURs

(Cont'd)

(2) $\Pi - II$: Unconditional Imitation with equal probability for best strategies (UI-2).

$$x_i(t+1) = x_{j^*_\mu}(t), \quad \text{with } p^i_\mu = \frac{1}{r}, \quad \mu = 1, \cdots, r. \tag{10}$$

$\Rightarrow$ Probabilistic k-valued dynamics.

(3) $\Pi - III$: Simplified Fermi Rule (FR).
Randomly (uniformly) choose a neighborhood $j \in U(i)$.

$$x_i(t+1) = \begin{cases} x_j(t), & c_j(x(t)) > c_i(x(t)) \\ x_i(t), & \text{otherwise.} \end{cases} \tag{11}$$

$\Rightarrow$ Probabilistic k-valued dynamics.

☞ Some SURs

(Cont'd)

(4) $\Pi - IV$: Myopic Best Response Adjustment (MBRA).

$$x_i(t+1) \in BR^i\left(x_{-i}(t)\right). \tag{12}$$

When $|BR^i| > 1$:

- Fixed Priority:
  $\Rightarrow$ Deterministic k-valued dynamics.
- Random Choice:
  $\Rightarrow$ Probabilistic k-valued dynamics.

III. STP and its Applications to Finite Games

☞ Semi-tensor Product (STP) of Matrices

Definition 3.8

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \operatorname{lcm}(n, p).$$

Then the right semi-tensor product of A and B is defined as

$$A \ltimes B := \left(I_{t/n} \otimes A\right) \left(I_{t/p} \otimes B\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \tag{13}$$

☞ Application of STP

- Analysis and control of logical systems;
- Finite game theory;
- Fuzzy system;
- Graph theory and formation control;
- Finite Automata;
- Coding;
- Engineering Systems: (i) Power systems, (ii) Mix-power vehicle, etc.

Authors of STP papers from:

- Over 40 Universities from China including Tsinghua Univ., Pekin Univ., Southeast Univ., Central South Univ. Nankai Univ, Tongji Univ. Harbin Institute of Technology, South China Univ. of Technology, etc.
- Countries: Italy, Israel, Japan, USA, UK, German, Australian, Russian, Sweden, South African, Singapore, India, Iran, Saudi Arabia, etc.

☞ Vector Form of Strategies

$$S_i = \mathcal{D}_k = \{1, 2, \cdots, k\}$$
$$\Downarrow$$
$$i \sim \delta_k^i$$
$$\Downarrow$$
$$S_i = \Delta_k = \{\delta_k^1, \delta_k^2, \cdots, \delta_k^k\},$$

where δ_k^i is the i-th column of I_k.

The set of mixed strategies

$$\bar{S}_i := \left\{ (r_1, \cdots, r_k)^T \mid r_j \geq 0, \sum_{j=1}^{k} r_j = 1 \right\},$$

where

$$(r_1, \cdots, r_k)^T \in \Upsilon_k.$$

☞ Strategy Profile Dynamics (SPD)

As long as the network graph (N, E) and the fundamental network game G are fixed: SUR $\Rightarrow$ SPD:

$$x_i(t+1) = g_i\left(x_j(t), c_j(t)\middle| j \in U(i)\right), \quad t \geq 0, \quad i \in N.$$

Since $c_j(t)$ depends on $x_k(t)$, $k \in U(j)$, it follows that

$$x_i(t+1) = f_i(x_j(t)\big| j \in U_2(i)), \quad i \in N. \tag{14}$$

$$x_i(t+1) = M_i \ltimes_{j \in U_2(i)} x_j(t) = \tilde{M}_i \ltimes_{j=1}^{n} x_j(t), \quad i \in N. \tag{15}$$

Then we have the strategy updating dynamics as

$$x(t+1) = Lx(t), \tag{16}$$

where $L = \tilde{M}_1 * \tilde{M}_2 * \cdots * \tilde{M}_n$.

☞ An Example

Example 3.12

Consider $((N,E),G,\Pi)$, where
Fundamental Network Game G is Rock-Scissors-Paper. The payoff bi-matrix is shown in Table 2.
Network Graph: R^3

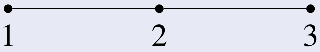

图 **3:** Network Graph R^3

Strategy Updating Rule:
$\Pi - I$: (unconditional imitation with fixed priority)

Example 3.12 (cont'd)

表 **3:** Payoffs $\to$ Dynamics

Profile	111	112	113	121	$\cdots$	333
C_1	0	0	0	1	$\cdots$	0
C_2	0	1	-1	-2	$\cdots$	0
C_3	0	-1	1	1	$\cdots$	0
f_1	1	1	1	1	$\cdots$	3
f_2	1	1	3	1	$\cdots$	3
f_3	1	1	3	1	$\cdots$	3

Example 3.12 (cont'd)

Identifying $1 \sim \delta_3^1$, $2 \sim \delta_3^2$, $3 \sim \delta_3^3$, we have the vector form of each f_i as

$$x_i(t+1) = f_i(x_1(t), x_2(t), x_3(t)) = M_i x_1(t)x_2(t)x_3(t), \quad i = 1, 2, 3, \tag{17}$$

where

$$\begin{aligned}
M_1 &= \delta_3[1\ 1\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 2\ 3\ 3\ 3];\\
M_2 &= \delta_3[1\ 1\ 3\ 1\ 1\ 1\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 1\ 2\ 2\ 3\ 2\ 3];\\
M_3 &= \delta_3[1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3].
\end{aligned}$$

Example 3.12 (cont'd)

Finally, we have the Strategy Profile Dynamics (SPD) as

$$x(t+1) = Lx(t), \tag{18}$$

where

$$L = M_1 * M_2 * M_3 = \delta_{27}[1\ 1\ 9\ 1\ \cdots, 27].$$

IV. Potential Game

☞ What Is a Potential Game?

Definition 4.1

Consider a finite game $G = (N, S, C)$. G is a potential game if there exists a function $P : S \to \mathbb{R}$, called the potential function, such that for every $i \in N$ and for every $s_{-i} \in S_{-i}$ and $\forall x, y \in S_i$

$$c_i(x, s_{-i}) - c_i(y, s_{-i}) = P(x, s_{-i}) - P(y, s_{-i}), \quad i = 1, \cdots, n. \tag{19}$$

D. Monderer, L.S. Shapley, Potential Games *Games and Economic Behavior*, Vol. 14, 124-143, 1996.

☞ Fundamental Properties

Theorem 4.2

If G is a potential game, then the potential function P is unique up to a constant number. Precisely if P_1 and P_2 are two potential functions, then $P_1 - P_2 = c_0 \in \mathbb{R}$.

Theorem 4.3

Every finite potential game possesses a pure Nash equilibrium. Sequential or cascading MBRA leads to a Nash equilibrium.

It becomes the kernel of Game theoretic control (Marden 2009).

J.R. Marden, G. Arslan, J. S. Shamma, Cooperative control and potential games, *IEEE Trans. Sys., Man, Cybernetcs, Part B*, Vol. 39, No. 6, 1393-1407, 2009.

Verify Potential Game

Shapley	$(96):$	$O(k^4)$	[6];	Hofbauer	$(02):$	$O(k^3)$	[7];
Hilo	$(11):$	$O(k^2)$	[8];	Cheng	$(14):$	min	[9].

Hilo: "It is not easy, however, to verify whether a given game is a potential game."

- D. Monderer, L.S. Shapley, Potential games, Games Econ. Theory, 97, 81-108, 1996.
- J. Hofbauer, G. Sorger, A differential game approach to evolutionary equilibrium selection, Int. Game Theory Rev. 4, 17-31, 2002.
- Y. Hino, An improved algorithm for detecting potential games, Int. J. Game Theory, 40, 199-205, 2011.
- D. Cheng, On finite potential games, *Automatica*, Vol. 50, No. 7, 1793-1801, 2014.

Lemma 4.4

G is a potential game if and only if there exist $d_i(x_1,\cdots,\hat{x}_i,\cdots,x_n)$, which is independent of x_i, such that

$$\begin{aligned} c_i(x_1,\cdots,x_n) = & P(x_1,\cdots,x_n) \\ & + d_i(x_1,\cdots,\hat{x}_i,\cdots,x_n), \quad i=1,\cdots,n, \end{aligned} \tag{20}$$

where P is the potential function.

Structure Vector Express:

$$\begin{array}{lll} c_i(x_1,\cdots,x_n) & := & V_i^c \ltimes_{j=1}^n x_j \\ d_i(x_1,\cdots,\hat{x}_i,\cdots,x_n) & := & V_i^d \ltimes_{j\neq i} x_j, \quad i=1,\cdots,n, \\ P(x_1,\cdots,x_n) & := & V_P \ltimes_{j=1}^n x_j. \end{array}$$

Construct:

$$\begin{aligned} E_i &= I_{k^{i-1}} \otimes \mathbf{1}_k \otimes I_{k^{n-i}} \\ &\in \mathcal{M}_{k^n \times k^{n-1}}, \quad i = 1, \cdots, n. \end{aligned} \tag{21}$$

$$\xi_i := \left(V_i^d\right)^T \in \mathbb{R}^{k^{n-1}}, \quad i = 1, \cdots, n. \tag{22}$$

$$b_i := (V_i^c - V_1^c)^T \in \mathbb{R}^{k^n}, \quad i = 2, \cdots, n. \tag{23}$$

☞ Potential Equation

Then (37) can be expressed as a linear system:

$$E\xi = b, \tag{24}$$

where

$$E = \begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}; \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}; \quad b = \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}. \tag{25}$$

(41) is called the potential equation and E is called the potential matrix.

☞ Main Result

Theorem 4.5

A finite game G is potential if and only if the potential equation has solution. Moreover, the potential P can be calculated by

$$V_P = V_1^c - V_1^d M_1 = V_1^c - \xi_1^T \left(\mathbf{1}_k^T \otimes I_k\right). \tag{26}$$

Example 4.6

Consider a prisoner's dilemma with the payoff bi-matrix as in Table 4.

表 **4:** Payoff Bi-matrix of Prisoner's Dilemma

$P_1 \backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

1: confess; 2: not confess

$$R = 5; \quad S = 0; \quad T = 10; \quad P = 1.$$

Example 4.6 (cont'd)

From Table 4

$$V_1^c = (R, S, T, P)$$
$$V_2^c = (R, T, S, P).$$

Assume $V_1^d = (a, b)$ and $V_2^d = (c, d)$. It is easy to calculate that

$$\begin{aligned} E_1 &= \left(D_f^{[2,2]}\right)^T = \delta_2[1,2,1,2]^T, \\ E_2 &= \left(D_r^{[2,2]}\right)^T = \delta_2[1,1,2,2]^T. \end{aligned}$$

$$b_2 = (V_2^c - V_1^c)^T = (0, T-S, S-T, 0)^T.$$

Example 4.6 (cont'd)

Then the potential equation (41) becomes

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ T-S \\ S-T \\ 0 \end{bmatrix}. \tag{27}$$

Example 4.6 (cont'd)

It is easy to solve it out as

$$\begin{cases} a = c = T - c_0 \\ b = d = S - c_0 \end{cases}$$

where $c_0 \in \mathbb{R}$ is an arbitrary number. We conclude that the general **Prisoner's Dilemma is a potential game**.

Using (26), the potential can be obtained as

$$\begin{aligned} V_P &= V_1^c - V_1^d D_f^{[2,2]} \\ &= (R - T, 0, 0, P - S) + c_0(1, 1, 1, 1). \end{aligned} \tag{28}$$

[Monderer, Shapley, 1996] considered the Prisoner's Dilemma with $R = 1$, $S = 9$, $T = 0$, $P = 6$, and $V_P = (4, 3, 3, 0)$. It is a special case of (28) with $c_0 = 3$.

☞ Potential NEG

Theorem 4.7

Consider an NEG, $((N, E), G, \Pi)$. If the fundamental network game G is potential, then the NEG is also potential. Moreover, the potential P of the NEG is:

$$P(s) := \sum_{(i,j) \in E} P^{i,j}(s_i, s_j). \tag{29}$$

Example 4.8

Consider an NEG $((N, E), G, \Pi)$, where the network graph is described as in Fig. 4.

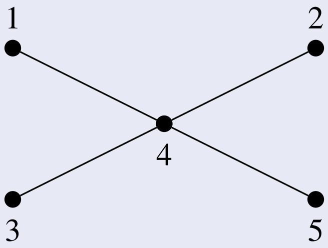

图 **4:** Network Graph

Example 4.8 (cont'd)

Assume:

- G: the prisoner's dilemma with $R = -1$, $S = -10$, $T = 0$, $P = -5$.
- Π: MBRA (Potential $\Rightarrow$ Pure Nash Equilibrium)

$$E = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{128\times 80}.$$

Example 4.8 (cont'd)

It is easy to check that

$$V_1^c = [-1 \quad -1 \quad -10 \quad -10 \quad -1 \quad -1 \quad -10 \quad -10$$
$$-1 \quad -1 \quad -10 \quad -10 \quad -1 \quad -1 \quad -10 \quad -10$$
$$0 \quad 0 \quad -5 \quad -5 \quad 0 \quad 0 \quad -5 \quad -5$$
$$0 \quad 0 \quad -5 \quad -5 \quad 0 \quad 0 \quad -5 \quad 5].$$

$$V_2^c = [-1 \quad -1 \quad -10 \quad -10 \quad -1 \quad -1 \quad -10 \quad -10$$
$$0 \quad 0 \quad -5 \quad -5 \quad 0 \quad 0 \quad -5 \quad -5$$
$$-1 \quad -1 \quad -10 \quad -10 \quad -1 \quad -1 \quad -10 \quad -10$$
$$0 \quad 0 \quad -5 \quad -5 \quad 0 \quad 0 \quad -5 \quad -5].$$

Example 4.8 (cont'd)

$$V_3^c = \begin{bmatrix} -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \end{bmatrix}.$$

$$V_4^c = \begin{bmatrix} -4 & -13 & 0 & -5 & -13 & -22 & -5 & -10 \\ -13 & -22 & -5 & -10 & -22 & -31 & -10 & -15 \\ -13 & -22 & -5 & -10 & -22 & -31 & -10 & -15 \\ -22 & -31 & -10 & -15 & -31 & -40 & -15 & -20 \end{bmatrix}.$$

Example 4.8 (cont'd)

$$V_5^c = \begin{array}{l}[-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5 \\ \;\;-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5 \\ \;\;-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5 \\ \;\;-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5].\end{array}$$

Using Theorem 3.5, we can check that the networked game is potential.

Example 4.8 (cont'd)

Moreover,

$$\xi_1 = \begin{bmatrix} 28 & 27 & 15 & 10 & 27 & 26 & 10 & 5 \\ 27 & 26 & 10 & 5 & 26 & 25 & 5 & 0 \end{bmatrix}.$$

Using (29), we have

$$V_P = \begin{bmatrix} -29 & -28 & -25 & -20 & -28 & -27 & -20 & -15 \\ -28 & -27 & -20 & -15 & -27 & -26 & -15 & -10 \\ -28 & -27 & -20 & -15 & -27 & -26 & -15 & -10 \\ -27 & -26 & -15 & -10 & -26 & -25 & -10 & -5 \end{bmatrix}.$$

Example 4.8 (cont'd)

Calculating P separately.
First, for any $(i,j) \in E$ we have

$$P(x_i, x_j) = V_0 x_i x_j, \tag{30}$$

where

$$V_0 = (R - T, 0, 0, P - S) = (-1\ 0\ 0\ 5).$$

Next, we have

$$\begin{aligned} V_P^{1,2} &= V_0 D_r^{[4,8]} = V_0 \left(I_4 \otimes \mathbf{1}_8^T\right) \\ &= [-1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \\ &\quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \\ &\quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \\ &\quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5 \quad\ \ 5]. \end{aligned}$$

Example 4.8 (cont'd)

Similarly, we can figure out all $V_P^{i,j}$ as

$$
\begin{aligned}
V_P^{1,3} &= V_0 D_r^{[2,2]} D_r^{[8,2]}, & V_P^{1,4} &= V_0 D_r^{[2,4]} D_r^{[16,2]}, \\
V_P^{1,5} &= V_0 D_r^{[2,8]}, & V_P^{2,3} &= V_0 D_f^{[2,2]} D_r^{[8,4]}, \\
V_P^{2,4} &= V_0 D_f^{[2,2]} D_r^{[4,2]} D_r^{[16,2]}, & V_P^{2,5} &= V_0 D_f^{[2,2]} D_r^{[4,4]} \\
V_P^{3,4} &= V_0 D_f^{[4,2]} D_r^{[16,2]}, & V_P^{3,5} &= V_0 D_f^{[4,2]} D_r^{[8,2]}, \\
V_P^{4,5} &= V_0 D_f^{[8,2]}.
\end{aligned}
$$

Example 4.8 (cont'd)

$$\begin{aligned} V_{\tilde{P}} &= V_P^{1,4} + V_P^{2,4} + V_P^{3,4} + V_P^{4,5} \\ &= \begin{bmatrix} -4 & -3 & 0 & 5 & -3 & -2 & 5 & 10 \\ -3 & -2 & 5 & 10 & -2 & -1 & 10 & 15 \\ -3 & -2 & 5 & 10 & -2 & -1 & 10 & 15 \\ -2 & -1 & 10 & 15 & -1 & 0 & 15 & 20 \end{bmatrix}. \end{aligned}$$

Comparing this result with the above V_P, one sees easily that

$$\tilde{P}(x) = P(x) + 25.$$

☞ Weighted Potential Game

Definition 4.9

Consider a finite game $G = (N, S, C)$. G is a weighted potential game if there exist a function $P : S \to \mathbb{R}$, called the potential function, and a set of weights $w_i > 0$, $i = 1, 2, \cdots, n$, such that for every $i \in N$ and for every $s_{-i} \in S_{-i}$ and $\forall x, y \in S_i$

$$c_i(x, s_{-i}) - c_i(y, s_{-i}) = w_i[P(x, s_{-i}) - P(y, s_{-i})], \quad i = 1, \cdots, n. \tag{31}$$

☞ Calculating Weights

Weighted Potential Equation

$$E_w \xi = B, \tag{32}$$

where

$$E_w = \begin{bmatrix} -w_2 E_1 & w_1 E_2 & 0 & \cdots & 0 \\ -w_3 E_1 & 0 & w_1 E_3 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ -w_n E_1 & 0 & 0 & \cdots & w_1 E_n \end{bmatrix}; \tag{33}$$

$$B = \begin{bmatrix} (w_1 V_2 - w_2 V_1)^T \\ (w_1 V_3 - w_3 V_1)^T \\ \vdots \\ (w_1 V_n - w_n V_1)^T \end{bmatrix}; \tag{34}$$

Weight Equation

$$\left[\left(I_{n-1} \otimes E_1\right) W_{[\kappa/k_1, n-1]} \xi_1 - \left(I_{n-1} \otimes V_1^T\right)\right] \begin{bmatrix} w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix}$$
$$= \begin{bmatrix} E_2 & 0 & \cdots & 0 \\ 0 & E_3 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & E_n \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{bmatrix} - \begin{bmatrix} V_2^T \\ V_3^T \\ \vdots \\ V_n^T \end{bmatrix}. \tag{35}$$

☞ Algorithm

:

Assume $w_i = 1$, $\forall i$;

(1)

$$\text{WP-Equation} \rightarrow \xi;$$

(2)

$$\text{W-Equation} \rightarrow \{w_i\};$$

back to (1).

D. Cheng, Z. Ji, Weighted and Near Weighted Potential Games with Application to Game Theoretic Control, (submitted)

IV. Decomposition of Finite Games

☞ Vector Space Structure of Finite Games

Denote by $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ the set of finite games with $|N| = n$, $|S_i| = k_i$, $i = 1, \cdots, n$. Then all games in $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ are distinct by their payoffs:

$$c_i(x_1, \cdots, x_n) = V_i^c \ltimes_{j=1}^n x_j, \quad i = 1, \cdots, n,$$

where $V_i^c \in \mathbb{R}^k$ ($k = \prod_{i=1}^n k_i$) is the structure vector of c_i. Set

$$V_G := [V_1^c, V_2^c, \cdots, V_n^c] \in \mathbb{R}^{nk}.$$

Then each V_G corresponds to a $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. Hence , $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ has a natural vector space structure as

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} \simeq \mathbb{R}^{nk}.$$

☞ Subspaces

Definition 4.1

Some subspaces of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ are defined as:

(i) Potential Games

(ii) Nonstrategic Games $G, \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is nonstrategic if

$$c_i(x_i, s_{-i}) = c_i(y_i, s_{-i}), \quad \forall x_i, y_i \in S_i,\ i = 1, \cdots, n. \tag{36}$$

(iii) $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is harmonic if

$$\sum_{i=1}^{n} \left(c_i(s) - \frac{1}{k_i} \sum_{x_i \in S_i} c_i(x_i, s_{-i}) \right) = 0. \tag{37}$$

Definition 4.1(cont'd)

(iv) G is pure harmonic, if

$$\sum_{i=1}^{n} c_i(s) = 0, \quad s \in S; \tag{38}$$

and

$$\sum_{x \in S_i} c_i(x, y) = 0, \quad \forall y \in S_{-i} := \prod_{j \neq i} S_j; \; i = 1, \cdots, n. \tag{39}$$

Example 4.2

Rock-Paper-Scissors is a pure harmonic game.

☞ Potential-Based Decomposition

Theorem 4.3

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{Harmonic\quad games}}_{Potential\quad games} \oplus \mathcal{H}. \qquad (40)$$

where $\mathcal{P}$, $\mathcal{N}$, $\mathcal{H}$ are pure potential games, nonstrategic games, and pure harmonic games respectively.

- Candogan et al[13]: Algebraic Topology, Graphic Decomposition, Weighted Orthogonality
- Cheng et al[14]: STP Approach, Euclidian Orthogonality.

- [13] O. Candogan, I. Menache, A. Ozdaglar, P.A. Parrilo, Flows and decompositions of games: Harmonic and potential games, *Mathematcs of Operations Research*, Vol. 36, No. 3, 474-503, 2011.
- [14] D. Cheng, T. Liu, K. Zhang, On decomposed subspaces of finite games, *IEEE Trans. Aut. Contr.*, Vol. 61, No. 11, 3651-3656, 2016.

☞ Symmetric-Based Decomposition

Theorem 4.4

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \mathcal{S} \oplus \mathcal{A} \oplus \mathcal{K}, \tag{41}$$

where $\mathcal{S}$, $\mathcal{A}$, $\mathcal{K}$ are symmetric, skew-symmetric, and asymmetric games respectively.

[15] Y. Hao, D. Cheng, On skew-symmetric games, *Journal of the Franklin Institute*, Vol. 355, 3196-3220, 2018.

☞ Relationship Between Potential and Symmetric

$$\text{Boolean} + \text{Symmetric} \Rightarrow \text{Potential}$$
$$\text{Boolean} + \text{Potential} \not\Rightarrow \text{Symmetric}$$

D. Cheng, T. Liu, From Boolean game to potential game, *Automatica*, Vol. 96, 51-60, 2018.

V. Bayesian Game

☞ Bayesian Game (Incomplete Information Game)

Definition 5.1

A (Finite) Static Bayesian Game $G(N, T, A, c, p)$ consists of

(i) Players:

$$N = \{1, 2, \cdots, n\}. \tag{42}$$

(ii) Types:

$$T = \{T_1, T_2, \cdots, T_n\}, \tag{43}$$

where

$$T_i = \{t_i^1, t_i^2, \cdots, t_i^{s_i}\}, \quad i = 1, 2, \cdots, n. \tag{44}$$

Definition 5.1(cont'd)

(iii) Profile-Actions:

$$A = A_1 \times A_2 \times \cdots \times A_n, \tag{45}$$

where

$$A_i = \left\{A_i^1, A_i^2, \cdots, A_i^{r_i}\right\}, \quad i = 1, 2, \cdots, n. \tag{46}$$

(iv) Payoff Functions:

$$c_i : (A_1 \times \cdots \times A_n) \times (t_1 \times \cdots \times t_n) \to \mathbb{R}, \quad i = 1, 2, \cdots, n. \tag{47}$$

(v) Pre-assigned Distribution:

$$Pr(t_1, t_2, \cdots, t_n). \tag{48}$$

Then the Beliefs (信念, 推断) is obtained as

$$\begin{aligned} p_{t_i} &:= Pr\left(t_{-i} \mid t_i\right) = \frac{Pr(t_1, t_2, \cdots, t_n)}{Pr(t_i)} \\ &= \frac{Pr(t_i, t_{-i})}{\sum\limits_{t_{-i}} Pr(t_i, t_{-i})}. \end{aligned} \tag{49}$$

Where $Pr(t_i, t_2, \cdots, t_n)$ is a common knowledge.

- R. Gibbons, *A Primer in Game Theory*, Prentice, London, 1992.
- D. Cheng, C. Li, Matrix Expression of Bayesian Game, submitted for pub.

☞ Vector Expression of Finite BG

Denote

$$\begin{array}{rcl} A_i & = & \left\{\delta_{r_i}^j \mid 1 \le j \le r_i\right\}; \\ T_i & = & \left\{\delta_{s_i}^j \mid 1 \le j \le s_i\right\}, \quad i = 1, 2, \cdots, n. \end{array}$$

$$c_i(a_1, \cdots, a_n; t_1, \cdots, t_n) \in \mathbb{R}, \quad a_j \in A_j(t_j), \quad 1 \le i \le n. \tag{50}$$

Denote by $r = \prod_{i=1}^n r_i$, $s = \prod_{i=1}^n s_i$, then we have

$$c_i : \Delta_{rs} \to \mathbb{R}, \quad 1 \le i \le n. \tag{51}$$

Using (50), it is obvious that for each i there exists a unique row vector $V_i \in \mathbb{R}^{st}$ such that

$$c_i = V_i t x, \quad 1 \leq i \leq n, \tag{52}$$

where $x = \ltimes_{i=1}^{n} x_i$, $t = \ltimes_{i=1}^{n} t_i$.

Remark 5.2

In fact, (52) provides a natural vector space structure for $c_i(x, t) \in \mathbb{R}^{st}$, where $x = a_1 a_2 \cdots a_n \in \Delta_r$ $t = t_1 t_2 \cdots t_n \in \Delta_s$.

☞ Two kinds of Types

Definition 5.4

There are two kinds of types:

(i) Types of Nature (TN):

The types are determined by pre-assigned distribution $Pr(t_1, t_2, \cdots, t_n)$, which is a common knowledge.
(Each player i knows the type t_i assigned to him.)

(ii) Types of Human (TH):

Player i has the right to choose t_i for entering the game.
(Then t_i becomes part of strategy for player i, $i = 1, 2, \cdots, n$.)

☞ Expected Payoff Function

Proposition 5.5

(i) TN:
The expected value of player i, using actions $a = (a_1, a_2, \cdots, a_n)$ is

$$e_i^N(a) := E_i(a(T)) = \sum_{t \in T} p_t V_i^c ta, \quad i = 1, 2, \cdots, n. \tag{53}$$

(ii) TH:
The expected value of player i, using actions $a = (a_1, a_2, \cdots, a_n)$ and type t_i^j is

$$\begin{aligned} e_i^H(a, t_i^j) &:= E_i(a(T) | t_i = t_i^j) \\ &= \sum_{t_{-i} \in T_{-i}} p_{t_i^j}(t_{-i}) V_i^c t(t_i^j, t_{-i}) a, \quad i = 1, 2, \cdots, n. \end{aligned} \tag{54}$$

☞ Bayesian-Nash Equilibrium

Definition 5.6

(i) TN:
A profile $(a_1^*, a_2^*, \cdots, a_n^*)$ is said to be a pure Bayesian-Nash equilibrium, if for each i and any $t_i^j \in T_i$ the following inequalities hold.

$$E_i^N(a^*(t)|t_i) \geq E_i^H\left(a_1^*(t_1), \cdots, a_{i-1}^*(t_{i-1}), a_i, .a_{i+1}^*(t_{i+1}), \cdots, a_n^*(t_n)|t_i\right), \quad \forall t_i \in T_i;\ i = 1, 2, \cdots, n. \tag{55}$$

(ii) TH:
A profile $(a_1^*(t_1^*), a_2^*(t_2^*), \cdots, a_n^*(t_n^*))$ is said to be a pure Bayesian-Nash equilibrium, if for each i and any $t \in T$ the following inequalities hold.

$$E_i^H\left(a_1^*(t_1^*), a_2^*(t_2^*), \cdots, a_n^*(t_n^*)\right) \geq E_i^H\left(a_1^*(t_1^*), \cdots, a_{i-1}^*(t_{i-1}^*), a_i(t_i), a_{i+1}^*(t_{i+1}^*), \cdots, a_n^*(t_n^*)\right), \forall t_i^j \in T_i,\ i = 1, 2, \cdots, n.$$

Definition 5.7

Assume the distribution $f(t_1, t_2, \cdots, t_n)$ is a common knowledge. Three conversions are defined as follows:

(i) (Harsanyi) H-Conversion: Define

$$c_i^H(a) := Ec_i(a), \quad i = 1, 2, \cdots, n. \tag{57}$$

(ii) (Selten) S-Conversion: Player i knows his type $t_i = t_i^\theta$. Define

$$c_i^S(a) := E(c_i(a)|t_i = t_i^\theta), \quad i = 1, 2, \cdots, n. \tag{58}$$

(iii) (Action-Type) AT-Conversion: Player i is able to choose t_i. Define

$$\begin{aligned} c_i^{AC}(t_i, a) \quad := \quad & \left[E(c_i(a)|t_i = t_i^1), E(c_i(a)|t_i = t_i^2),\right. \\ & \left.\cdots, E(c_i(a)|t_i = t_i^{s_i}),\right], \quad i = 1, 2, \cdots, n. \end{aligned} \tag{59}$$

VI. Game Theoretic Control

☞ Framework of Game Theoretic Control [14]

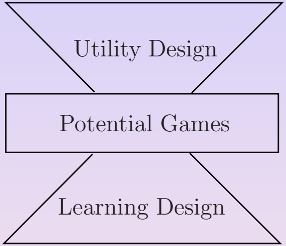

图 **5:** 'Hourglass' Architecture of Game Theoretic Control

[14] R. Gopalakrishnan, J.R. Marden, A. Wierman, An architectural view of game theoretic control, *ACM SIGMETRICS*, Vol. 38, No. 3, 31-36, 2010.

☞ Utility (Cost Function) Design

Congestion Game

Definition 6.1

A congestion game, denoted by $C\,(N, M, (\mathcal{A}^i)_{i\in N}, (\Xi_j),\ j \in M)$, consists of

- Player: $N = \{1, 2, \cdots, n\}$;
- Facility: $M = \{1, 2, \cdots, m\}$;
- Facility Cost: $\Xi_j(k)$, $j \in M$, (depends on number of users k);
- Strategy: $\mathcal{A}^i \subset 2^M$, Facilities used by player i, $i = 1, \cdots, n$.

Denote

- Profile: $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}^i$;
- Number of Users:

$$r_j(a) = \left|\{i \mid j \in a_i\}\right|, \quad j \in M. \tag{60}$$

- Utilities:

$$c_i(a) = \sum_{j \in a_i} \Xi_j(r_j(a)), \quad i \in N. \tag{61}$$

Theorem 6.2

A congestion game is a potential game with potential function, $P : A \to \mathbb{R}$, as

$$P(a) = \sum_{j \in \cup_{i=1}^{n} a_i} \left(\sum_{\ell=1}^{r_j(a)} \Xi_j(\ell) \right). \tag{62}$$

D. Monderer, L. Shapley, Potential games, *Game and Economic Behavior*, Vol. 14, 124-143, 1996.

☞ Semi-tensor Product Expression

- Facility Const:

$$\Xi_j = \left[\xi_1^j, \xi_2^j, \cdots, \xi_n^j\right], \quad j \in M.$$

$$\Xi := [\Xi_1, \Xi_2, \cdots, \Xi_m] \in \mathbb{R}^{mn}. \tag{63}$$

- Strategy:

$$a_i = [a_i^1, \cdots, a_i^m], \quad i \in N,$$

where

$$a_i^j = \begin{cases} 1, & \text{Player } i \text{ uses } j \\ 0, & \text{Player } i \text{ does not use } j \end{cases}$$

☞ Semi-tensor Product Expression (cont'd)

- Index Number:

$$\alpha(a)=\sum_{i=1}^{n} a_i \in \mathbb{R}^m, \quad a\in \mathcal{A}.$$

Set

$$d_i(a)=\begin{cases}\delta_n^{\alpha_i}, & \alpha_i\neq 0\\ \mathbf{0}_n, & \text{Otherwise.}\end{cases} \tag{64}$$

☞ Utilities

$$D(a) := \mathrm{diag}\left(d_1(a), d_2(a), \cdots, d_n(a)\right) \tag{65}$$

Proposition 6.3

Utilities can be expressed as

$$c_i(a) = \Xi D(a) a_i, \quad i = 1, \cdots, n. \tag{66}$$

☞ Potential Function

- Index Vector:

$$b_i(a) := [\underbrace{1, \cdots, 1}_{\alpha_i(a)}, \underbrace{0, \cdots, 0}_{n-\alpha_i(s)}]$$

$$i = 1, 2, \cdots, n.$$

Set

$$B(a) = [b_1(a), b_2(a), \cdots, b_n(a)]. \tag{67}$$

Proposition 6.4

The potential function is

$$P(a) = \Xi B^T(a). \tag{68}$$

☞ Facility-based System

Definition 6.5

1. A facility-based system, denoted as $\Sigma = (M, N, (\mathcal{A}^i)_{i\in N}, P)$, where $M = \{1, 2, \cdots, m\}$ is the set of facilities, $N = \{1, 2, \cdots, n\}$ set of users, $\mathcal{A}^i \subset 2^M$, the set of strategies of player i.
2. $\mathcal{A} := \prod_{i=1}^{n} \mathcal{A}^i$ is the profile.
3. $P : \mathcal{A} \to \mathbb{R}$ is the global cost function.

- Problem I: Find $a^* \in \mathcal{A}$, such that

$$P(a^*) = \min_{a\in\mathcal{A}} P(a). \tag{69}$$

☞ Problem I $\Rightarrow$ Problem II

- Problem II: Design facility cost to turn the problem into a congestion game.

Let $|\mathcal{A}| = \ell$, Set $\mathcal{A} = \{A_1, A_2, \cdots, A_\ell\}$. Construct

$$B\Xi^T = P, \tag{70}$$

where

$$B = \begin{bmatrix} B(A_1) \\ B(A_2) \\ \vdots \\ B(A_\ell) \end{bmatrix}; \quad P = \begin{bmatrix} P(A_1) \\ P(A_2) \\ \vdots \\ P(A_\ell) \end{bmatrix},$$

$B(a)$ is constructed as in (67).

☞ Main Result

Theorem 6.6

Given a facility-based system. It can be converted via designed facility cost functions into a congestion game with a global cost $P(a)$ as its potential function, if and only if, equation (70) has solution. Moreover, the solution is the facility cost functions.

Y. Hao, S. Pan, Y. Qiao, D. Cheng, Cooperative control vial congestion game, *IEEE TAC*, Vol. 63, No. 12, 4361-4366，2018.

Remark 6.7

1. A proper algorithm, say MBRA, makes it converging to a pure Nash equilibrium.
2. Unique Nash equilibrium $\Rightarrow$ minimum value.

☞ Learning Design

Method 1: Joint Strategy Fictitious Play(JSFP)

$$\begin{cases} p_i^{a_i(t-1)}(t) = \epsilon, \\ p_i^{a_i^*}(t) = 1 - \epsilon., \end{cases}$$

where $0 < \epsilon < 1$ is the inertia,

$$V_i^{a_i}(t) = \frac{1}{t} \sum_{\tau=1}^{t-1} u_i(a_i, a_{-i}(\tau)).$$

$$a_i^* \in \operatorname{argmax}_{a_i \in \mathcal{A}_i} V_i^{a_i}(t).$$

- The global behavior converges almost surely to a pure Nash equilibrium.

Method 2: Log-linear learning

$$p_i^{a_i}(t) = \frac{e^{\frac{1}{T}u_i(a_i,a_{-i}(t-1))}}{\sum\limits_{a_i' \in \mathcal{A}_i} e^{\frac{1}{T}u_i(a_i',a_{-i}'(t-1))}},$$

where $T \geq 0$ is a temperature coefficient.

- The global behavior converges to a stationary distribution

$$\mu^a = \frac{e^{\frac{1}{T}P(a)}}{\sum\limits_{a' \in \mathcal{A}} e^{\frac{1}{T}P(a')}},$$

  which maximizes the potential function $P(s)$.

C. Li, Y. Xing, F. He, D. Cheng, A strategic learning algorithm for state-based games, Automatica, Volume 113, 2020, 108615, ISSN 0005-1098, https://doi.org/10.1016/j.automatica.2019.108615.

VII. Concluding Remarks

☞ General Remark

- Game theory is a wide branch of mathematics. There are many open problems for further study.
- Game theoretic control is a growing new cross discipline direction.
- STP seems to be a powerful tool for finite games.

☞ Some Topics For Further Study (by STP)

- Verifying continuous potential game.
- Learning algorithm design for game theoretic control.
- Dynamics and control of networked Bayesian (incomplete information) game.
- $\cdots$

Thank you for your attention!

Question?