Control Theory vs Game Theory

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Outline of Presentation

I. Control Theory Compared with Game Theory

☞ Control Theory

Figure 1: Norbert Wiener

N. Wiener, *Cybernetics, or Control and Communication in the Animal and the Machine*, Hermann & Camb. Press, Paris, 1948.

☞ Game Theory

Figure 2: John von Neumann

J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944.

An Introduction to Game Theory

☞ 1. (Normal Form) Non-cooperative Games

Definition 1.1

A normal non-cooperative game $G = (N, S, c)$:

(i) **Player**: $N = \{1, 2, \cdots, n\}$.

(ii) **Strategy**:

$$S_i = \{1, 2, \cdots, k_i\}, \quad i = 1, \cdots, n;$$

Situation (Profile): $S = \prod_{i=1}^{n} S_i$.

(iii) **Payoff function**:

$$c_j(s) : S \to \mathbb{R}, \quad j = 1, \cdots, n. \qquad (1)$$

Payoff:

$$c = \{c_1, \cdots, c_n\}.$$

☞ Nash Equilibrium

Definition 1.2

In a normal game G, a situation

$$s = (x_1^*; \quad ; x_n^*)\ 2\ S$$

is a Nash equilibrium if

$$c_j(x_1^*; \quad ; ; x_j^*; \quad ; x_n^*) \quad c_j(x_1^*; \quad ; x_j; \quad ; x_n^*) \\ j = 1; \quad ; n: \tag{2}$$

Example 1.3

Consider a game G with two players: P_1 and P_2:

- Strategies of P_1: $D_2 = f1;2g$;
- Strategies of P_2: $D_3 = f1;2;3g$.

Table 1: Payoff bi-matrix

$P_1 n P_2$	1	2	3
1	$2;\ 1$	$3;\ 2$	6, 1
2	$1;\ 6$	$2;\ 3$	$5;5$

Nash Equilibrium is $(1;2)$.

☞ 2. Cooperative Game

Definition 1.4

A (transferable utility) game G consists of three ingredients:

(i) n players $N := \{p_1, \cdots, p_n\} = \{1, \cdots, n\}$;

(ii) subsets $\{S | S \in 2^N\}$, each S is called a coalition; $S = \emptyset$ is empty coalition, $S = N$ is complete coalition.

(iii) $v : 2^N \to \mathbb{R}$ is called the characteristic function; $v(S)$ is the worth of S, (which means the profit (cost: $c : 2^N \to \mathbb{R}$) of coalition S).

$$v(\emptyset) = 0.$$

Example 1.5 (Glove Game)

Consider a game G with $P = \{p_1, p_2, \ldots, p_n\}$.

$$R = \{p_i \in P | p_i \text{ has a right hand glove}\}$$

$$L = \{p_i \in P | p_i \text{ has a left hand glove}\}$$

Let $S \in 2^P$. A singe glove (0.01), a pair of gloves (1), then:

$$v(S) = \min\{|S \cap L|, |S \cap R|\} + 0.01\,[n - 2\min\{|S \cap L|, |S \cap R|\}].$$

☞ Imputation

Definition 1.6

Given a cooperative game $G = (N, v)$.

- $x \in \mathbb{R}^n$ is called an imputation, if

$$x_i \geq v(\{i\}), \quad i = 1, \dots, n, \tag{3}$$

$$\sum_{i=1}^{N} x_i = v(N). \tag{4}$$

☞ 3. Evolutionary (Non-cooperative) Games

Assumptions:

(i) finitely or infinitely repeated:

$$G \to G^N, \quad \text{or} \quad G \to G^\infty$$

(ii) Dynamics of strategies:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), \cdots, x_1(1), \cdots, x_n(1)) \\ x_2(t+1) = f_2(x_1(t), \cdots, x_n(t), \cdots, x_1(1), \cdots, x_n(1)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), \cdots, x_1(1), \cdots, x_n(1)), \end{cases} \tag{5}$$

where $x_i \in \mathcal{D}_{k_i}$, and $f_i : \prod_{j=1}^n \mathcal{D}_{k_j}^t \to \mathcal{D}_{k_i}$, $i = 1, \cdots, n$.

☞ 4. Networked Evolutionary Game

Definition 1.7

•

☞ evolution ! cooperation

- Cooperation based on reciprocity can get started in an asocial world.
- Cooperation in organisms has been a difficulty for evolutionary theory since Darwin.

[1] R. Axelrod, W.D. Hamilton, *The Evolotion of Cooperation*, Science, New York, 1981.

[2] M.A. Nowak, Five rules for the evolution of cooperation, Science, 314: 1560-1563, 2006.

[3] D.Okada, P.M. Bingham, Human uniqueness-self-interest and social cooperation, *J. Theor. Biol.*, Vol. 253, No. 2, 261-270, 2008.

Control Compared with Game

☞ Common Point: the purpose of actions

An individual intends to "manipulate" the object.

☞ Different Point:

Object:

- (for control) Machine (not intelligent);
- (for game) Intelligent object (ability in anti-control).

Goal:

- (for control) Optimization;
- (for game) Nash Equilibrium.

Example 1.8

Control:

Consider a linear system

$$\dot{x} = Ax + Bu; \tag{6}$$

the problem is to minimize J,

$$\min_u J := \min_u \int_0^\infty \left[x^T Q x + u^T R u \right] \, dt: \tag{7}$$

The optimal control is:

$$u^* = \quad R^{-1} B^T P x; \tag{8}$$

where $P \quad 0$ satisfying Algebraic Riccati Equation:

$$PA + A^T P = Q \quad PBR^{-1} B^T P = 0: \tag{9}$$

Example 1.8(cont'd)

Game:

Consider a linear system

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2; \tag{10}$$

where u_i is to minimize J_i, $i = 1; 2$,

$$\min_{u_i} J_i := \min_{u_i} \int_0^\infty \left[x^T Q_i x + u^T R_i u \right] \, dt; \quad i = 1; 2: \tag{11}$$

Example 1.8(cont'd)

The Nash equilibrium is

$$\begin{cases} u_1^* = \quad R_1^{-1} B_1^T P_1 x; \\ u_2^* = \quad R_2^{-1} B_2^T P_2 x; \end{cases} \tag{12}$$

where $P_i > 0$, $i = 1; 2$, satisfying coupled Algebraic Riccati Equations:

$$\begin{cases} P_1(A \quad B_2 R_2^{-1} B_2^T P_2) + (A \quad B_2 R_2^{-1} B_2^T P_2)^T P_1 \\ \quad +Q_1 \quad P_1 B_1 R_1^{-1} B_1^T P_1 = 0: \\ P_2(A \quad B_1 R_1^{-1} B_1^T P_1) + (A \quad B_1 R_1^{-1} B_1^T P_1)^T P_2 \\ \quad +Q_2 \quad P_2 B_2 R_2^{-1} B_2^T P_2 = 0: \end{cases} \tag{13}$$

As for nonlinear case it becomes a problem of Differential Games, and we have coupled Hamilton-Jacobi-Bellman equation.

- J.C. Engwerda, Computational aspects of the open-loop Nash equilibrium in linear quadratic games, *J. Econ. Dyn. Contr.*, Vol. 22, No. 8-9, 1487-1506, 1998.
- A. Friedman, *Differential Games*, American Math. Society, Rhode Island, 1974.
- N.Y. Lukoyanov, A Hamilton-Jacobitype equation in control problems with hereditory information, *J. Appl. Math. Mech.*, Vol. 64, No. 2, 243-253, 2000.
- F.L. Lewis, et al, *Optimal Control*, John Wiley& Sons, New Jersey, 2012.

☞ Cross Discipline between Control and Game

- Control Theory) Game Theory:
  Control-orient Games

- Game Theory) Control Theory:
  Game-based Controls

II. Control-oriented Games

II.1. Learning Control in Games

Strategy in Rock-Paper-Scissors Game

MIT Best 50 of 2014 from Social Science: How to win in Rock-Paper-Scissors?

Z. Wang, B. Xu, H. Zhou, *Social cycling and conditional responses in the Rock-Paper-Scissors Game*, Scientific Reports 4, Vol. 5830, 2014.

Convolutional Neural Network *!* Optimal Action

V. Mnih, et al (19), *Human-level control through deep reinforcement learning*, Nature, Vol. 518, 529-533, 2015.

☞ Different viewpoints to games

Example 2.1

Consider: Rock-Paper-Scissors

- Game Theory (God's perspective):

  The game properties: Zero-sum; Pure harmonic game; Nash equilibrium: $(1=3; 1=3; 1=3)$;

- Control Theory (Player's perspective):

  How to win?

  Successes depending on the whole knowledge about all related aspects (知己知彼, 百战不殆)

Example 2.1(cont'd)

Frequency) Strategy:

F: opponent's strategy frequency:

$$F = (f_r; \ f_s; \ f_p) \ \ ) \ \ x = \frac{1}{f_r + f_s + f_p}(f_r; \ f_s; \ f_p)$$

Assume: $F(0) := (1; \ 1; \ 1)$. Then

$$F(t+1) = \begin{cases} (f_r(t) + 1; f_s(t); f_p(t)); & x(t) = r \\ (f_r(t); f_s(t) + 1; f_p(t)); & x(t) = s \\ (f_r(t); f_s(t); f_p(t) + 1); & s(t) = p; \end{cases}$$

where $x(t)$ is the opponent's strategy at t. Then

$$u(t+1) \ 2 \ BR(x(t)):$$

II. Control-oriented Games

II.2. State-Space Approach

☞ Networked Evolutionary Game

Definition 1.7(recall)

A networked evolutionary game, denoted by $((N;E);G;\Pi)$, consists of

(i) a network (graph) $(N;E)$;

(ii) an FNG, G, such that if $(i;j)$ 2 E, then i and j play FNG with strategies $x_i(t)$ and $x_j(t)$ respectively;

(iii) a local information based strategy updating rule.

☞ Network Graph

Definition 2.2

1. $(N; E)$ is called a graph, where N is the set of nodes and $E \quad N \quad N$ is the set of edges.
2.

   $U_d(i) =$ fjjthere is a path connecting $i; j$ with leng $\quad dg$

3. If $(i; j)$ 2 E implies $(j; i)$ 2 E the graph is undirected, otherwise, it is directed.

Definition 2.3

A network is homogeneous network, if each node has same degree (for undirected graph)/ in-degree and out-degree(for directed graph).

☞ Fundamental Network Game

Definition 2.4

(i) A normal game with two players is called a fundamental network game (FNG), if

$$S_1 = S_2 := S_0 = \{1, 2, \dots, k\}.$$

(ii) An FNG is symmetric, if

$$c_{1,2}(x, y) = c_{2,1}(y, x), \quad \forall x, y \in S_0.$$

☞ Overall Payoff

$$c_i(t) = \sum_{j \in U(i) \setminus i} c_{ij}(t), \quad i \in N. \tag{14}$$

☞ Strategy Updating Rule

Definition 2.5

A strategy updating rule (SUR) for an NEG, denoted by Π, is a set of mappings:

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \,\big|\, j \in U(i)\}\right), \quad t \geq 0, \quad i \in N. \tag{15}$$

Remark 2.6

1. f_i could be a probabilistic mapping;
2. When the network is homogeneous, f_i, $i \in N$, are the same.

Example 2.6

- Π I: *Unconditional Imitation with fixed priority*:

$$j^* = \operatorname{argmax}_{j \in U(i)} c_j(x(t)); \qquad (16)$$

)

$$x_i(t+1) = x_{j^*}(t): \qquad (17)$$

In non-unique case:

$$\operatorname{argmax}_{j \in U(i)} c_j(x(t)) := f j_1^*; \quad ; j_r^* g;$$

set priority:

$$j^* = \min f \; j \;\; 2 \operatorname{argmax}_{j \in U(i)} c_j(x(t)) g: \qquad (18)$$

) Deterministic k-valued dynamics.

Example 2.6(cont'd)

- Π II: *Unconditional Imitation with equal probability for best strategies*.

$$x_i(t+1) = x_{j^*_\mu}(t); \quad \text{with } \; p^i_\mu = \frac{1}{r}; \qquad = 1; \qquad ; r: \tag{19}$$

) Probabilistic k-valued dynamics.

- Π III: *Simplified Fermi Rule*. Randomly choose a neighborhood j 2 $U(i)$.

$$x_i(t+1) = \begin{cases} x_j(t); & c_j(x(t)) > c_i(x(t)) \\ x_i(t); & \text{Otherwise}: \end{cases} \tag{20}$$

) Probabilistic k-valued dynamics.

☞ Fundamental Evolutionary Equation

Recall SUR (15):

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \big| j \in U(i)\}\right), \quad t \geq 0, \quad i \in N.$$

Since $c_j(t)$ depends on $x_k(t)$, $k \in U(j)$, it follows that $x_i(t+1)$ depends on $x_j(t)$, $j \in U_2(i)$. That is, we can rewrite (15) as

$$x_i(t+1) = f_i(\{x_j(t) \big| j \in U_2(i)\}), \quad i \in N. \tag{21}$$

Remark 2.7

(i) Using the SUR, the f_i, $i \in N$ can be determined. Then (21) is called the FEE.

(ii) For a homogeneous network all f_i are the same.

☞ Calculating FEE

Example 2.8

Consider Rock - Scissors - Cloth on R_3. The payoff bi-matrix is:

Table 2: Payoff Bi-matrix (Rock-Scissors-Cloth)

$P_1 n P_2$	$R = 1$	$S = 2$	$C = 3$
$R = 1$	$(0;\ 0)$	$(1;\quad 1)$	$(\quad 1;\ 1)$
$S = 2$	$(\quad 1;\ 1)$	$(0;\ 0)$	$(1;\quad 1)$
$C = 3$	$(1;\quad 1)$	$(\quad 1;\ 1)$	$(0;\ 0)$

Assume the strategy updating rule is $\Pi \quad I$:

Example 2.9 (cont'd)

Table 3: Payoffs ! Dynamics

Profile	111	112	113	121	122	123
C_1	0	0	0	1	1	1
C_2	0	1/2	-1/2	-1	-1/2	0
C_3	0	-1	1	1	0	-1
f_1	1	1	1	1	1	1
f_2	1	1	3	1	1	1
f_3	1	1	3	1	2	2
Profile	131	132	133	211	212	213
C_1	-1	-1	-1	-1	-1	-1
C_2	1/2	1	0	1	0	1/2
C_3	0	-1	1	-1	1	0
f_1	1	1	1	3	3	3
f_2	1	1	3	3	2	3
f_3	1	1	3	3	2	3

Example 2.9 (cont'd)

Profile	221	222	223	231	232	233
C_1	0	0	0	1	1	1
C_2	-1/2	0	1/2	0	-1	-1/2
C_3	1	0	-1	-1	1	0
f_1	2	2	2	2	2	2
f_2	1	2	2	2	2	2
f_3	1	2	2	3	2	3
Profile	311	312	313	321	322	323
C_1	1	1	1	-1	-1	-1
C_2	-1/2	0	-1	0	1/2	1
C_3	0	-1	1	1	0	-1
f_1	3	3	3	2	2	2
f_2	3	3	3	1	2	2
f_3	1	1	3	1	2	2

Example 2.9 (cont'd)

Profile	331	332	333
C_1	0	0	0
C_2	1/2	-1/2	0
C_3	-1	1	0
f_1	3	3	3
f_2	3	2	3
f_3	3	2	3

Example 2.9 (cont'd)

Identifying $1 \quad ^{1}_{3}$, $2 \quad ^{2}_{3}$, $3 \quad ^{3}_{3}$, we have the vector form of each f_i as

$$x_i(t+1) = f_i(x_1(t); x_2(t); x_3(t)) = M_i x_1(t) x_2(t) x_3(t); \quad i = 1; 2; 3; \tag{22}$$

where

$$\begin{aligned} M_1 &= {}_3[1\ 1\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 2\ 3\ 3\ 3]; \\ M_2 &= {}_3[1\ 1\ 3\ 1\ 1\ 1\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 1\ 2\ 2\ 3\ 2\ 3]; \\ M_3 &= {}_3[1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3]: \end{aligned}$$

Example 2.9 (cont'd)

Assume the strategy updating rule is $\Pi \quad II$:
Since player one and player 3 have no choice, f_1 and f_3 are the same as in Π is BNS. That is,

$$M_1' = M_1; \quad M_3' = M_3:$$

Consider player 2, who has two choices: either choose 1 or choose 3, and each choice has probability $0:5$. Using similar procedure, we can finally figure out f_2 as:

Example 2.9 (cont'd)

$$
M_2' = \left[\begin{array}{cccccccccccccc}
1 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & 1
\end{array}\right]
$$

Now the evolution dynamics becomes a probabilistic 3-valued logical network. (to be completed!)

☞ Further Investigations

- Convergence of NEG;
- Strategically equivalence;
- Evolutionarily stable strategy (ESS).

D.Cheng, F. He, H. Qi, T. Xu, Modeling, analysis, and control of networked evolutionary games, *IEEE Trans. Aut. Contr.*, (**Regular Paper**), On line: DOI:10.1109/TAC.2015.2404471.

D.Cheng, T. Xu, H. Qi, Evolutionarily stable strategy of networked evolutionary games, *IEEE TNNLS*, Vol. 25, No. 7, 1335-1345, 2014 (**regular paper**).

D.Cheng, H. Qi, et al, Semi-tensor product approach to networked evolutionary games, *Contr. Theory Tech.*, Vol. 12, No. 2, 198-214, 2014.

II. Control-oriented Games

II.3. Man-Machine Games

Model (n machines vs m players)

$$\begin{cases} m_1(t+1) = f_1(m_1(t); \quad ; m_n(t); h_1(t); \quad ; h_m(t)) \\ m_2(t+1) = f_2(m_1(t); \quad ; m_n(t); h_1(t); \quad ; h_m(t)) \\ \vdots \\ m_n(t+1) = f_n(m_1(t); \quad ; m_n(t); h_1(t); \quad ; h_m(t)): \end{cases} \tag{23}$$

Goal:

$$\max_{h(t) \in \mathcal{D}_p} \sum_{t=1}^{N} \quad {}^{t} c_h(t); \quad 0 < \quad < 1: \tag{24}$$

- Pure Strategy
  The optimal solution appears on a cycle. Find best cycle.

  [1] Y. Mu, L. Guo, Optimization and identification in nonequilibrium dynamical games, *Proc. 48th IEEE CDC*, 5750-5755, 2009.

  [2] Y. Zhao, Z. Li, D. Cheng, Optimal control of logical control notworks, *IEEE Trans. Aut. Contr.*, Vol. 56, No. 8, 1766-1776, 2011 (**Regular Paper**).

- Mixed Strategy
  $N < \infty$: Dynamic Programming (DP)
  $N = \infty$: DP + Receding horizon control

  [1] D. Cheng, Y. Zhao, T. Xu, Receding horizon based feedback optimization for mix-valued logical networks, *IEEE Trans. Aut. Contr.*, On line: DOI:10.1109/TAC.2015.2419874.

III. Potential Games

Definition 3.1

Consider a finite game $G = (N, S, C)$. G is a positive game if there exists a function $P : S \to \mathbb{R}$, called the potential function, such that for every $i \in N$ and for every $s^{-i} \in S^{-i}$ and $\forall x, y \in S_i$

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = P(x, s^{-i}) - P(y, s^{-i}), \quad i = 1, \ldots, n. \tag{25}$$

D. Monderer, L.S. Shapley, Potential Games *Games and Economic Behavior*, Vol. 14, 124-143, 1996.

Fundamental Properties

Theorem 3.2

If G is a potential game, then the potential function P is unique up to a constant number. Precisely if P_1 and P_2 are two potential functions, then $P_1 - P_2 = c_0 \in \mathbb{R}$.

Theorem 3.3

Every finite potential game possesses a pure Nash equilibrium. Sequential or cascading MBRA leads to a Nash equilibrium.

☞ Verify Potential Game

- Shapley (96): $O(k^4)$;
- Hofbauer (02): $O(k^3)$;
- Hilo (11): $O(k^2)$;
- Cheng (14): Potential Equation.

Hilo: "It is not easy, however, to verify whether a given game is a potential game."

D. Monderer, L.S. Shapley, Potential games, Games Econ. Theory, 97, 81-108, 1996.

J. Hofbauer, G. Sorger, A differential game approach to evolutionary equilibrium selection, Int. Game Theory Rev. 4, 17-31, 2002.

Y. Hino, An improved algorithm for detecting potential games, Int. J. Game Theory, 40, 199-205, 2011.

D. Cheng, On finite potential games, *Automatica*, Vol. 50, No. 7, 1793-1801, 2014 (**regular paper**).

Lemma 3.4

G is a potential game if and only if there exist $d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n)$, which is independent of x_i,) such that

$$\begin{aligned} c_i(x_1, \cdots, x_n) &= P(x_1, \cdots, x_n) \\ &+ d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n), \quad i = 1, \cdots, n, \end{aligned} \tag{26}$$

where P is the potential function.

Structure Vector Express:

$$\begin{aligned} c_i(x_1, \cdots, x_n) &:= V_i^c \ltimes_{j=1}^n x_j \\ d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) &:= V_i^d \ltimes_{j \neq i} x_j, \quad i = 1, \cdots, n, \\ P(x_1, \cdots, x_n) &:= V_P \ltimes_{j=1}^n x_j. \end{aligned}$$

Construct:

$$
\begin{aligned}
\Psi_i &= I_{k^{i-1}} \quad \mathbf{1}_k \quad I_{k^{n-i}} \\
& 2 \ M_{k^n \times k^{n-1}}; \quad i = 1; \quad ; n: \qquad (27)
\end{aligned}
$$

$$
{}_i := \left(V_i^d\right)^T \ 2 \ \mathbb{R}^{k^{n-1}}; \quad i = 1; \quad ; n: \qquad (28)
$$

$$
b_i := (V_i^c \quad V_1^c)^T \ 2 \ \mathbb{R}^{k^n}; \quad i = 2; \quad ; n: \qquad (29)
$$

Potential Equation

Then (26) can be expressed as a linear system:

$$\Psi \quad = b; \tag{30}$$

where

$$\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 & 0 & & 0 \\ \Psi_1 & 0 & \Psi_3 & & 0 \\ \vdots & & & \ddots & \\ \Psi_1 & 0 & 0 & & \Psi_n \end{bmatrix}; \qquad = \begin{bmatrix} {}_1 \\ {}_2 \\ \vdots \\ {}_n \end{bmatrix}; \quad b = \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}: \tag{31}$$

(30) is called the potential equation and Ψ is called the potential matrix.

Main Result

Theorem 3.5

A finite game G is potential if and only if the potential equation has solution. Moreover, the potential P can be calculated by

$$V_P = V_1^c \quad V_1^d M_1 = V_1^c \quad {}_1^T \left(\mathbf{1}_k^T \quad I_k\right): \tag{32}$$

Example 3.6

Consider a prisoner's dilemma with the payoff bi-matrix as in Table 4.

Table 4: Payoff Bi-matrix of Prisoner's Dilemma

P_1nP_2	1	2
1	$(R;\ R)$	$(S;\ T)$
2	$(T;\ S)$	$(P;\ P)$

Example 3.6 (cont'd)

From Table 4

$$V_1^c = (R; S; T; P)$$
$$V_2^c = (R; T; S; P):$$

Assume $V_1^d = (a; b)$ and $V_2^d = (c; d)$. It is easy to calculate that

$$\Psi_1 = \left(D_f^{[2,2]}\right)^T = {}_2[1; 2; 1; 2]^T;$$
$$\Psi_2 = \left(D_r^{[2,2]}\right)^T = {}_2[1; 1; 2; 2]^T:$$

$$b_2 = (V_2^c \quad V_1^c)^T = (0; T \quad S; S \quad T; 0)^T:$$

Example 3.6 (cont'd)

Then the potential equation (30) becomes

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ T \quad S \\ S \quad T \\ 0 \end{bmatrix} : \tag{33}$$

Example 3.6 (cont'd)

It is easy to solve it out as

$$\begin{cases} a = c = T \quad c_0 \\ b = d = S \quad c_0 \end{cases}$$

where c_0 2 $\mathbb{R}$ is an arbitrary number. We conclude that the general **Prisoner's Dilemma is a potential game**. Using (32), the potential can be obtained as

$$\begin{aligned} V_P &= V_1^c \quad V_1^d D_f^{[2,2]} \\ &= (R \quad T; 0; 0; P \quad S) + c_0(1; 1; 1; 1): \end{aligned} \tag{34}$$

(Monderer, Shapley 1996) considered the Prisoner's Dilemma with $R = 1$, $S = 9$, $T = 0$, $P = 6$, and $V_P = (4; 3; 3; 0)$. It is a special case of (34) with $c_0 = 3$.

IV. Game-based Controls

☞ Challenges for Systems and Control in the 21st Century

Peter E. Caines at ICARCV, Dec. 2014, Singapore

S& C Challenge V: The development of a dynamic games theory (GT) of the formation and stability of coalitions.

- No tractable form yet in GT or economics.

☞ Game-based control in TAC

- G. Arslan, M.F. Demirkol, S. Yuksel, On games with coupled constraineds, IEEE Trans. Aut. Contr., Vol. 60, No. 2, 358-372, 2015.
- A. Cortes, S. Martinez, Self-triggered best-response dynamics for continuous games, IEEE Trans. Aut. Contr., Vol. 60, No. 4, 1115-1120, 2015.
- T. Mylvaganam, M. Sassano, A. Astolfi, Constructive e-Nash equilibria for nonzero-sum differatial games, IEEE Trans. Aut. Contr., Vol. 60, No. 4, 950-965, 2015.
- A. Nedic, D. Bauso, Dynamic coalitional TU games. distributed bargaining among players' neighbors, IEEE Trans. Aut. Contr., Vol. 58, No. 6, 1363-1376, 2013.

☞ Engineering Game Theory

S.W. Mei, F. Liu, Y. Wei

- Control of power systems via game theory;
- Multi-objective optimization via game theory;
- Robust optimization/control via game theory.

(**Merge control theory into game theory?**)

Game-based Controls

4.1 Consensus of MAS

- Network graph: $(N, E(t))$: $N = \{1, 2, \cdots, n\}$ with varying topology: $E(t)$.
- Model of MAS:

$$a_i(t+1) = f_i\left(a_j(t) \mid j \in U(i)\right), \quad i = 1, \cdots, n. \tag{35}$$

- Set of Strategies:

$$a_i \in A_i \subset \mathbb{R}^n, \quad i = 1, \cdots, n.$$

J.R. Marden, G. Arslan, J. S. Shamma, Cooperative control and potential games, *IEEE Trans. Sys., Man, Cybernetcs, Part B*, Vol. 39, No. 6, 1393-1407, 2009.

Potential Game Structure

- Potential Function:

$$P(a) = \sum_{i \in N} \sum_{j \in U(i)} \frac{k a_j \quad a_i k}{2}: \tag{36}$$

- Payoff Functions:

$$c_i(a) = \sum_{j \in U(i)} k a_j \quad a_i k; \quad i = 1; \quad ; n: \tag{37}$$

Remark 4.1.1

$$\max_{a \in \mathcal{A}} P(a) \ ) \ \text{Consensus}$$

Spatial Adaptive Player (SAP)

Taking a mixed strategy: The probability for a_i 2 A_i is:

$$r^{a_i}(t) = \frac{\exp f \quad c_i(a_i; a^{-i}(t \quad 1))g}{\sum_{\xi_i \in A_i} \exp f \quad c_i(\quad _i; a^{-i}(t \quad 1))g}. \tag{38}$$

- Stationary distribution:

$$(a) = \frac{\exp f \quad P(a)g}{\sum_{\xi \in A} \exp f \quad P(\quad)g}.$$

- As ! 1 , mixed strategies maximize the potential function.
- With sufficiently large , the players will asymptotically reach a consensus with arbitrarily high probability.

Restricted Spatial Adaptive Player (RSAP) with Binary LLL Algorithm

- Restricted Action Set:

$$R_i(a_i(t-1)) \subset A_i.$$

- Choosing $\hat{a}_i \in R_i(a_i(t-1))$:

$$\begin{aligned} Pr\left[\hat{a} = a_i\right] &= 1/z_i, \; a_i \neq a_i(t-1) \\ Pr\left[\hat{a} = a_i(t-1)\right] &= 1 - \tfrac{|R_i(a_i(t-1))|-1}{z_i}, \end{aligned}$$

where $z_i = \max_{a_i \in A_i} |R_i(a_i)|$.

- Mixed Strategy:

$$\begin{aligned} Pr\left[a_i(t) = \hat{a}_i\right] &= \tfrac{\exp\{\beta c_i(\hat{a}_i, a^{-i}(t-1))\}}{D} \\ Pr\left[a_i(t) = a_i(t-1)\right] &= \tfrac{\exp\{\beta c_i(a(t-1))\}}{D}, \end{aligned}$$

where

$$D = \exp\{\beta c_i(\hat{a}_i, a^{-i}(t-1))\} + \exp\{\beta c_i(a(t-1))\}.$$

Main Result

Assumptions:

1. (Reversibility): For $a_i^1; \ a_i^2 \ 2 \ A_i$,

$$a_i^1 \ 2 \ R_i(a_i^2) \quad , \quad a_i^1 \ 2 \ R_i(a_i^2):$$

2. (Feasibility): For $a_i^o; \ a_i^d \ 2 \ A_i$, there exists a sequence of actions $a_i^o \ ! \ a_i^1 \ ! \qquad ! \ a_i^d$ that satisfies $a_i^s \ 2 \ R_i(a_i^{s-1})$.

Theorem 4.1.2

Consider system (35). Assume 1 and 2, then BLLL induces the unique stationary distribution (38).

As long as is sufficiently large, a consensus will be reached with arbitrarily high probability.

4.2 Distributed Coverage of Graphs

Problem Statement

- Unknown connected graph $\mathtt{G} = (V, E)$.
- Mobile agents $N = \{1, 2, \dots, n\}$ (initially arbitrarily deployed on $\mathtt{G}$).
- Agent a_i can cover $U^i(t) := U_{d_i}(a_i(t))$, $i = 1, \dots, n$.

Purpose: $\max_a \bigcup_{i=1}^n U^i$.

- A.Y. Yazicioglu, M. Egerstedt, J.S. Shamma, A game theoretic approach to distributed coverage of graphs by heterogeneous mobile agents, *Est. Contr. Netw. Sys.*, Vol. 4, 309-315, 2013.
- M. Zhu, S. Martinez, Distributed coverage games for energy-aware mobile sensor networks, *SIAM J. Cont. Opt.*, Vol. 51, No. 1, 1-27, 2013.

Potential Game Formulation

- Potential Function:

$$P(a) = \sum_{i=1}^{n} \left| \bigcup_{i=1}^{n} U^i \right| : \tag{39}$$

- Payoff Functions:

$$c_i(a) = \left| U^i \mathrm{n} \bigcup_{j \neq i} U^j \right| : \tag{40}$$

Description

Restricted Action Set:

$$R_i(a_i(t \quad 1)) \quad V:$$

Assumptions: A1: Reversibility
A2: Feasibility
Region Condition:

- Covering radius d_j; (If $\quad 2\ U_{d_j}(j)$, then $\quad$ is covered j.)
- Communication radius d_j^c, (If $\quad 2\ U_{d_j^c}(j)$, the $U_{d_\xi}(\quad)$ is known by j.)

$$d_j^c \quad d_j \quad d^* + 1; \quad j = 1; \quad\quad ; n; \tag{41}$$

where

$$d^* = \max_{1 \leq j \leq n} d_j:$$

Main Result

Theorem 4.2.1

Assuming A1, A2, and (41) and using the BLLL algorithm (with large enough), the number of covered nodes is asymptotically maximized (in probability).

4.3 Congestion Games

An Example

Problem: Player 1 want to go from A to C, player 2 want to go from B to D:

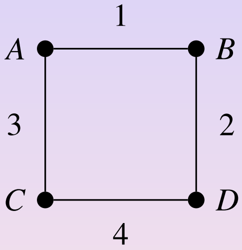

Figure 3: A Road Map

D. Monderer, L.S. Shapley, Potential Games, *Games & Economic Behavior*, Vol. 14, 124-143, 1996.

Example 4.3.1

Consider Figure 3.

- Player 1 his set of strategies is

$$S_1 = f1 \quad 2; \ 3 \quad 4g;$$

- Player 2 his set of strategies is

$$S_2 = f1 \quad 3; 2 \quad 4g:$$

The cost for road j to be used by s cars is denoted by $c_j(s)$.

Table 5: Payoff Bi-matrix of Roads

$P_1 n P_2$	$1 \quad 3$	$2 \quad 4$
$1 \quad 2$	$f_1(2)+f_2(1),\ f_1(2)+f_3(1)$	$f_2(2)+f_1(1),\ f_2(2)+f_4(1)$
$3 \quad 4$	$f_3(2)+f_4(1),\ f_3(2)+f_1(1)$	$f_4(2)+f_3(1),\ f_4(2)+f_2(1)$

Example 4.3.1(cont'd)

It is easy to verify that this is a potential game with

$$P(s_1; s_2) = \begin{cases} f_1(1) + f_1(2) + f_2(1) + f_3(1); & 1 \quad 2; 1 \quad 3 \\ f_2(1) + f_2(2) + f_1(1) + f_4(1); & 1 \quad 2; 2 \quad 4 \\ f_3(1) + f_3(2) + f_4(1) + f_1(1); & 3 \quad 4; 1 \quad 3 \\ f_4(1) + f_4(2) + f_3(1) + f_2(1); & 3 \quad 4; 2 \quad 4: \end{cases}$$

Congestion Model

Definition 4.3.2

A congestion model $C\,(N; M; (\Sigma^i)_{i\in N}; (f_j)_{j\in M})$ is defined as follows.

- Players: $N = f1; 2; \quad ; ng$;
- Facilities: $M = f1; 2; \quad ; mg$;
- Set of strategies: $S_i := \Sigma_i \quad 2^M$;
- Facility cost: $f_j : N \;!\; R$ (depends on number of users).

Let A_i 2 Σ_i be a strategy.

$$A := \prod_{i=1}^{n} A_i \; 2 \; \Sigma := \prod_{i=1}^{n} \Sigma_i:$$

For each $j \in M$ Set

$$\sigma_j(A) := \#\{i \in N \mid j \in A_i\}.$$

Then we define

- Payoff Functions:

$$c_i(A) := \sum_{j \in A_i} f_j\left(\sigma_j(A)\right). \tag{42}$$

- Potential Function:

$$P(A) := \sum_{j \in \cup_{i=1}^{n} A_i} \left(\sum_{\ell=1}^{\sigma_j(A)} f_j(\ell) \right). \tag{43}$$

Main Results

Theorem 4.3.3

Every congestion game is a potential game.

Theorem 4.3.4

Every finite potential game is isomorphic to a congestion game.

An example to road pricing [1].

X. Wang, N. Xiao, T. Wongpiromsarn, L. Xie, E. Frazzoli, D. Rus, Distributed consensus in noncooperative congestion games: an application to road pricing, *Proc. 10th IEEE Int. Conf. Contr. Aut.*, Hangzhou, China, 1668-1673, 2013.

4.4 Some Other Related Topics

- Scheduling-Allocation (in Power Systems)

T. Heikkinen, A potential game approach to distributed power control and scheduling, *Computer Networks*, Vol 50, 2295-2311, 2006.

R. Bhakar, V.S. Sriram, et al, Probabilistic game approaches for network cost allocation, *IEEE Trans. Power Sys.*, Vol. 25, No. 1, 51-58, 2010.

- Cooperative Game *!* Control

A. Nedic, D. Bauso, Dynamic coalitional TU gemes: distributed bargaining among players' neighbors, *IEEE TAC*, Vol. 58, No. 6, 1363-1376, 2013.

V. Conclusion

1. Game Theory and Control Theory are deeply inter-connected.
2. Control Theory ! Game Theory:
    - State Space approach;
    - Human-machine games;
    - Learning control in games.
3. Game Theory ! Control Theory (Potential):
    - Control of MASs via designed potentials;
    - Distributed graph covering;
    - Congestion control;
    - Control of power systems, etc.

A cross discipline between Control Theory and Game Theory is emerging!

Thank you for your attention!

Question?