From STP to Logical Dynamic Systems

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Outline of Presentation

I. Semi-tensor Product of Matrices

☞ Tensor (Kronecker) Product

$$A_{m\times n} \otimes B_{p\times q} :=$$

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{11}B & a_{12}B & \cdots & a_{1m}B \\ \vdots & & & \\ a_{11}B & a_{12}B & \cdots & a_{1m}B \end{bmatrix} \in \mathcal{M}_{mp\times nq}.$$

☞ An Example

Example 1.1

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad B = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$A \otimes B = \begin{bmatrix} a & 0 & b & 0 & c & 0 \\ 0 & a & 0 & b & 0 & c \\ d & 0 & e & 0 & f & 0 \\ 0 & d & 0 & e & 0 & f \end{bmatrix}.$$

☞ Semi-tensor Product of Matrices

$A_{m\times n} \times B_{p\times q} =?$

Definition 1.2

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \operatorname{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \tag{1}$$

☞ Some Basic Comments

- When $n = p$, A n $B = AB$. So the STP is a generalization of conventional matrix product.
- When $n = rp$, denote it by $A \succ_r B$;
  when $rn = p$, denote it by $A \prec_r B$.
  These two cases are called the multi-dimensional case, which is particularly important in applications.
- STP keeps almost all the major properties of the conventional matrix product unchanged.

☞ Examples

Example 1.3

1. Let $X = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$X \textcolor{red}{\text{n}}\ Y = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 7 & 0 \end{bmatrix}.$$

2. Let $X = \begin{bmatrix} -1 & 2 & 1 & -1 & 2 & 3 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}$. Then

$$X \textcolor{red}{\text{n}}\ Y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 2 + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot (-2) = \begin{bmatrix} -3 \\ -6 \end{bmatrix}.$$

Example 1.3 (Continued)

3. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$A \text{ n } B = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

☞ Insight Meaning

Let $A \in \mathcal{M}_{m\times n}$. Consider a bilinear form

$$P(x, y) = x^T Ay. \tag{2}$$

Row Stacking Form:

$$V_r(A) = (a_{11}, a_{12}, \cdots, a_{1n}, \cdots, a_{m1}, \cdots, a_{mn}).$$

Column Stacking Form

$$V_c(A) = (a_{11}, a_{21}, \cdots, a_{m1}, \cdots, a_{1n}, \cdots, a_{mn}).$$

Then (using Row Stacking Form:)

$$P(x, y) = V_r(A) \text{ n } x \text{ n } y. \tag{3}$$

n can search pointer mechanically!

☞ Multi-linear Mapping

$$P : \mathrm{R}^m \times \mathrm{R}^n \times \mathrm{R}^s \to \mathrm{R}.$$

Cubic Matrix?

$$\begin{matrix} d_{111} & d_{112} & \cdots & d_{11n} \\ d_{121} & d_{122} & \cdots & d_{12n} \\ \vdots & \vdots & & \vdots \\ d_{1m1} & d_{1m2} & \cdots & d_{1mn} \end{matrix}$$

$$\begin{matrix} d_{k11} & d_{k12} & \cdots & d_{k1n} \\ d_{k21} & d_{k22} & \cdots & d_{k2n} \\ \vdots & \vdots & & \vdots \\ d_{km1} & d_{km2} & \cdots & d_{kmn} \end{matrix}$$

kth layer

$$\begin{matrix} d_{p11} & d_{p12} & \cdots & d_{p1n} \\ d_{p21} & d_{p22} & \cdots & d_{p2n} \\ \vdots & \vdots & & \vdots \\ d_{pm1} & d_{pm2} & \cdots & d_{pmn} \end{matrix}$$

$$
\begin{aligned}
& P(\delta_m^i, \delta_n^j, \delta_s^k) := d_{i,j,k}, \\
& \quad i = 1, \cdots, m; j = 1, \cdots, n; k = 1, \cdots, s.
\end{aligned}
$$

Define

$$
M_P = [d_{111}, \cdots, d_{11s}, \cdots, d_{mn1}, \cdots, d_{mns}].
$$

Then

$$
P(x, y, z) = M_P \;\mathrm{n}\; x \;\mathrm{n}\; y \;\mathrm{n}\; z. \tag{4}
$$

It is available for general multi-linear mappings.

☞ A Syntheses

$$
\begin{array}{rcl}
STP & : & A_{m\times n} \text{ n } B_{p\times q} \\
\\
n = p & \to & AB = A \text{ n } B \text{ (Conventional)} \\
A_i := \mathrm{Col}_i(A) & \to & A \otimes B = [A_1 \text{ n } B, \cdots, A_n \text{ n } B] \text{ (Kronecker)} \\
n = q & \to & A * B = [A_1 \text{ n } B_1, \cdots, A_n \text{ n } B_n] \text{ (Khatri-Lao)}
\end{array}
$$

- a syntheses of multi-products;
- with multi-functions of several products.

☞ Properties

Proposition 1.4

- (Distributive rule)

$$\begin{aligned} &A \text{ n } (\alpha B + \beta C) = \alpha A \text{ n } B + \beta A \text{ n } C; \\ &(\alpha B + \beta C) \text{ n } A = \alpha B \text{ n } A + \beta C \text{ n } A, \quad \alpha, \beta \in \mathrm{R}. \end{aligned} \tag{5}$$

- (Associative rule)

$$A \text{ n } (B \text{ n } C) = (A \text{ n } B) \text{ n } C. \tag{6}$$

Proposition 1.5

- $$(A \text{ n } B)^T = B^T \text{ n } A^T. \tag{7}$$

- Assume both A and B are invertible. Then

$$(A \text{ n } B)^{-1} = B^{-1} \text{ n } A^{-1}. \tag{8}$$

Proposition 1.6 (Pseudo-Commutativity)

Assume $A \in \mathcal{M}_{m\times n}$ is given.

- Let $Z \in \mathrm{R}^t$ be a row vector. Then

$$A \;\mathrm{n}\; Z = Z \;\mathrm{n}\; (I_t \otimes A); \tag{9}$$

- Let $Z \in \mathrm{R}^t$ be a column vector. Then

$$Z \;\mathrm{n}\; A = (I_t \otimes A) \;\mathrm{n}\; Z. \tag{10}$$

☞ Multi-dimensional Cases

- Let $\xi \in \mathrm{R}^n$ be a column (row). Then

$$\xi^k := \underbrace{\xi \,\mathrm{n} \cdots \mathrm{n}\, \xi}_{k}.$$

- Let $A \in \mathcal{M}_{m\times n}$ and $m|n$ or $n|m$. Then

$$A^k := \underbrace{A \,\mathrm{n} \cdots \mathrm{n}\, A}_{k}.$$

- In Boolean algebra, all matrices $A \in \mathcal{M}_{m\times n}$, where $m = 2^p$ and $n = 2^q$ (or for k-valued case: $m = k^p$ and $n = k^q$), which is the multiple-dimensional case.

☞ Swap Matrix

Definition 1.7

A swap matrix, $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn)$ and its rows by $(11, 21, \cdots, m1, \cdots, 1n, 2n, \cdots, mn)$. Then its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

When $m = n$ we briefly denote $W_{[n]} := W_{[n,n]}$.

☞ Example

Example 1.8

Let $m = 2$ and $n = 3$, the swap matrix $W_{[2,3]}$ is constructed as

$$W_{[2,3]} = \begin{array}{c} \begin{array}{cccccc} (11) & (12) & (13) & (21) & (22) & (23) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{array}.$$

☞ Properties

Proposition 1.9

- Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X, \quad W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y. \tag{12}$$

- Let $A \in \mathcal{M}_{m\times n}$. Then

$$W_{[m,n]} V_r(A) = V_c(A), \quad W_{[n,m]} V_c(A) = V_r(A). \tag{13}$$

- Let $X_i \in \mathbb{R}^{n_i}$, $i = 1, \cdots, m$. Then

$$\begin{aligned} &\left(I_{n_1+\cdots+n_{k-1}} \otimes W_{[n_k,n_{k+1}]} \otimes I_{n_{k+2}+\cdots+n_m}\right) \\ &X_1 \ltimes \cdots \ltimes X_k \ltimes X_{k+1} \ltimes \cdots \ltimes X_m \\ &= X_1 \ltimes \cdots \ltimes X_{k+1} \ltimes X_k \ltimes \cdots \ltimes X_m. \end{aligned} \tag{14}$$

☞ Properties

Proposition 1.10

- The swap matrix is an orthogonal matrix as

$$W_{[m,n]}^{T} = W_{[m,n]}^{-1} = W_{[n,m]}. \tag{15}$$

-

$$W_{[m,n]} = \left[\delta_n^1 \text{ n } \delta_m^1 \quad \cdots \quad \delta_n^n \text{ n } \delta_m^1 \quad \cdots\cdots \quad \delta_n^n \text{ n } \delta_m^m\right], \tag{16}$$

where δ_n^i is the ith column of I_n.

☞ “$\times$” vs “n”

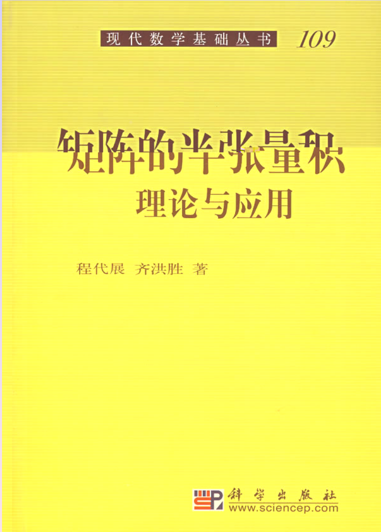
	CP $\times$	STP n
Domain	Equal Dimension	Arbitrary
Property	Similar	Similar
Applicability	linear, bilinear	multilinear
Commutativity	No	Pseudo-Commutative

Remark: Compare scalar product with matrix product:

- $a \times b$ is always defined $\Leftrightarrow$ $A \times B$ may not defined;
- $a \times b = b \times a$ $\Leftrightarrow$ in general $AB \neq BA$.

n overcomes these two obstacles!

☞ My Book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Communications and Control Engineering
Daizhan Cheng
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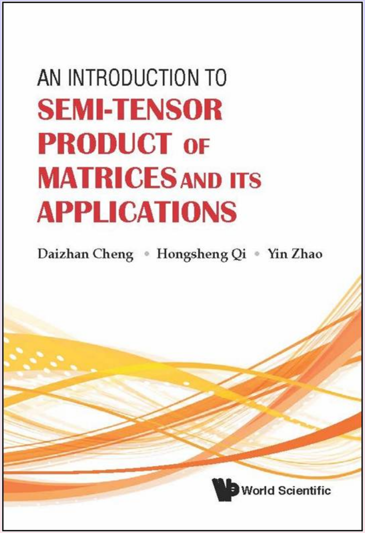
Analysis and Control of Boolean Networks
Semi-tensor Product Approach
Springer



AN INTRODUCTION TO
SEMI-TENSOR PRODUCT OF MATRICES AND ITS APPLICATIONS
Daizhan Cheng • Hongsheng Qi • Yin Zhao
World Scientific

II. Matrix Expression of Logic

☞ Logic

- $\mathcal{D} = \{0 \sim \text{False}, 1 \sim \text{True}\}$.

Logical Variables

$$x, y \cdots \in \mathcal{D}$$

Truth Table of Logical Functions

Table 1: Negation ($\neg x$)

x	$\neg x$
1	0
0	1

☞ Logic (continued)

Truth Table of Logical Functions (continued)

Table 2: Disjunction: $(x \vee y)$; Conjunction: $(x \wedge y)$; Conditional: $(x \rightarrow y)$; Biconditional: $(x \leftrightarrow y)$; Exclusive Or: $(x \bar{\nabla} y)$.

x	y	$x \vee y$	$x \wedge y$	$x \rightarrow y$	$x \leftrightarrow y$	$x \bar{\nabla} y$
1	1	1	1	1	1	0
1	0	1	0	0	0	1
0	1	1	0	1	0	1
0	0	0	0	1	1	0

☞ Vector Form of Logic

- δ_n^i: the ith column of I_n;
- $\Delta_n := \{\delta_n^i | i = 1, \cdots, n\}$, $\Delta := D_2$;

$$\text{True} \sim 1 \sim \delta_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$\text{False} \sim 0 \sim \delta_2^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- A matrix $L \in \mathcal{M}_{n\times r}$ is called a logical matrix if

$$\mathrm{Col}(L) \subset \Delta_n.$$

Denote by $\mathcal{L}_{n\times r}$ the set of $n \times r$ logical matrices.

- Let $L = [\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_r}] \in \mathcal{L}_{n\times r}$. Briefly,

$$L = \delta_n[i_1, i_2, \cdots, i_r].$$

Example 2.1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \delta_3[1,3,2,3].$$

☞ Vector Form of Logical Mapping

$$1 \sim \delta_2^1; \text{ and } 0 \sim \delta_2^2 \Rightarrow \mathcal{D} \sim \Delta.$$

Hence,

- Logical function:

$$f : \mathcal{D}^n \to \mathcal{D} \quad \Rightarrow \quad \Delta^n \to \Delta;$$

- Logical mapping:

$$F : \mathcal{D}^n \to \mathcal{D}^m \quad \Rightarrow \quad \Delta^n \to \Delta^m.$$

The later function (mapping) is called the vector form.

☞ Structure Matrix (1)

Theorem 2.2

Let $y = f(x_1, \cdots, x_n) : \Delta^n \to \Delta$. Then there exists unique $M_f \in \mathcal{L}_{2\times 2^n}$ such that

$$y = M_f x, \quad \text{where } x = \text{n}_{i=1}^{n} x_i. \tag{17}$$

Definition 2.3

The M_f is called the **structure matrix** of f.

☞ Structure Matrix (2)

Theorem 2.4

Let $F : \Delta^n \to \Delta^k$ be defined by

$$y_i = f_i(x_1, \cdots, x_n).$$

Then there exists unique $M_F \in \mathcal{L}_{2^k \times 2^n}$ such that

$$y = M_F x, \tag{18}$$

where

$$x = \ltimes_{i=1}^n x_i; \qquad y = \ltimes_{i=1}^k y_i.$$

Definition 2.5

The M_F is called the **structure matrix** of F.

☞ Structure Matrices of Logical Operators

Table 3: Structure Matrices of Logical Operators

$\neg$	M_n	$\delta_2[2\ 1]$
$\vee$	M_d	$\delta_2[1\ 1\ 1\ 2]$
$\wedge$	M_c	$\delta_2[1\ 2\ 2\ 2]$
$\rightarrow$	M_i	$\delta_2[1\ 2\ 1\ 1]$
$\leftrightarrow$	M_e	$\delta_2[1\ 2\ 2\ 1]$
$\bar{\vee}$	M_p	$\delta_2[2\ 1\ 1\ 2]$

An Example

Example 2.6

There are three persons.

- A said: “B is a liar!”
- B said: “C is a liar!”
- C said: “A and B both are liars!”

Who is the liar?

Set P: A is honest; Q: B is honest; R: C is honest.
The logical expression is

$$(P \leftrightarrow \neg Q) \wedge (Q \leftrightarrow \neg R) \wedge (R \leftrightarrow \neg P \wedge \neg Q) = 1.$$

Its matrix form is

$$L(P, Q, R) = M_c M_c (M_e P M_n Q)(M_e Q M_n R)(M_e R M_c M_n P M_n Q)$$

We can calculate the canonical form of $L(P, Q, R)$ as

$$L(P, Q, R) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} PQR = \delta_2^1.$$

Only if $P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then L is true, which means that only B is honest.

☞ Multi-valued Logic

- $\mathcal{D}_k = \{1, \frac{k-2}{k-1}, \cdots, \frac{1}{k-1}, 0\}$;
- $\Delta_k = \{\delta_k^1, \delta_k^2, \cdots, \delta_k^{k-1}, \delta_k^k\}$.

k-valued logical variables:

$$x, y \in \mathcal{D}_k$$

Using equivalence:

$$\delta_k^1 \sim 1, \quad \delta_k^2 \sim \frac{k-2}{k-1}, \quad , \cdots, \delta_k^k \sim 0,$$

we have

$$x, y \in \Delta_k.$$

Theorem 2.7

Let $y = f(x_1, \cdots, x_n) : \Delta_k^n \to \Delta_k$. Then there exists unique $M_f \in \mathcal{L}_{k \times k^n}$ such that

$$y = M_f x, \quad \text{where } x = \mathrm{n}_{i=1}^n x_i. \tag{19}$$

Example 2.8

A detective is investigating a murder case. He has the following clues:

1. 80% that A or B is the murderer;
2. If A is the murderer, the killing time is before midnight;
3. If B's confession is true, the light in the room of murder was on at the midnight;
4. If B's confession is a lie, it is very possible that the murder happened before midnight;
5. There is an evidence that the light in the room of murder at the midnight was off.

Example 2.8 (Continued)

Set $D_6 = \{T, \text{very likely}, 80\%, 1\text{-}80\%, \text{very unlikely}, F\}$.

- A: A is murderer;
- B: B is murderer;
- M: murder happened before midnight;
- S: B's confession is true;
- L: the light was on at midnight.

$$A \vee B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \tag{20}$$

$$A \to \neg M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{21}$$

$$S \to L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{22}$$

$$\neg S \to M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{23}$$

$$\neg L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{24}$$

Example 2.8 (Continued)

From (24) $\Rightarrow L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$
Then from (22), we have

$$M_i^6 SL = (M_i^6 W_{[6]} L)S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$\Rightarrow S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

Similarly, (23) $\Rightarrow M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$
Then from (21) $\Rightarrow A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$
Finally, from (20) $\Rightarrow B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$

We conclude that: A is **very unlikely** the murderer; B is **80%** the murderer.

III. Boolean Network

Kaffman: for cellular networks, gene regulatory networks, etc.

☞ Network Graph

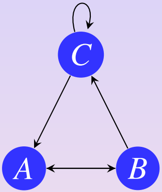


Figure 2: A Boolean network

☞ Network Dynamics

$$\begin{cases} A(t+1) = B(t) \wedge C(t) \\ B(t+1) = \neg A(t) \\ C(t+1) = B(t) \vee C(t) \end{cases} \tag{25}$$

Boolean Control Network

☞ Network Graph

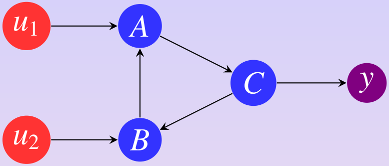


Figure 3: A Boolean control network

☞ Network Dynamics

Its logical equation is

$$\begin{cases} A(t+1) = B(t) \wedge u_1(t) \\ B(t+1) = C(t) \vee u_2(t) \\ C(t+1) = A(t) \end{cases} \qquad (26)$$
$$y(t) = \neg C(t)$$

☞ Dynamics of Boolean Network

$$
\begin{cases}
x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\
\vdots \\
x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \quad x_i \in \mathcal{D},
\end{cases}
\tag{27}
$$

where

$$
\mathcal{D} := \{0, 1\}.
$$

☞ Dynamics of Boolean Control Network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \end{cases}$$
$$y_j(t) = h_j(x(t)), \quad j = 1, \cdots, p, \tag{28}$$

where $x_i, u_i, y_i \in \mathcal{D}$.

☞ Matrix Expression of Subspace

- State Space: $\mathcal{X} = F_\ell(x_1, \cdots, x_n)$
- Subspace: $\mathcal{V} = F_\ell(y_1, \cdots, y_k)$, $y_i \in \mathcal{X}$ is described by

$$y_i = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, k.$$

- Algebraic Form:

$$y = F_v x,$$

  where

$$x = \mathrm{n}_{i=1}^{n} x_i, \ y = \mathrm{n}_{i=1}^{k} y_i, \ F_v \in \mathcal{L}_{2^k \times 2^n}.$$

- Conclusion: Each $F_v \in \mathcal{L}_{2^k \times 2^n}$ uniquely determines a subspace $\mathcal{V}$.

☞ Algebraic Form of BN (27)

$$x(t+1) = Lx(t), \tag{29}$$

where $L \in \mathcal{L}_{2^n \times 2^n}$.

☞ Algebraic Form of BCN (28)

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases} \tag{30}$$

where $L \in \mathcal{L}_{2^n \times 2^{n+m}}$, $H \in \mathcal{L}_{2^p \times 2^n}$.

Examples

Example 3.5

- Consider Boolean network (25) in Fig. 2. We have

$$L = \delta_8[3\ 7\ 7\ 8\ 1\ 5\ 5\ 6].$$

- Consider Boolean control network (26) in Fig. 3. We have

$$\begin{aligned} L &= \delta_8[1\ 1\ 5\ 5\ 2\ 2\ 6\ 6\ 1\ 3\ 5\ 7\ 2\ 4\ 6\ 8 \\ &\quad\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 5\ 7\ 5\ 7\ 6\ 8\ 6\ 8]; \\ H &= \delta_2[2\ 1\ 2\ 1\ 2\ 1\ 2\ 1]. \end{aligned}$$

☞ Topological Structure

- Find “fixed points”, “cycles”;
- Find “basin of attraction” ,“transient time”;
- “Rolling Gear” structure, which explains why “tiny attractors” decide “vast order”.

References:

D. Cheng, H. Qi, A linear representation of dynamics of Boolean networks, *IEEE Trans. Aut. Contr.*, vol. 55, no. 10, pp. 2251-2258, 2010. (**Regular Paper**)

D. Cheng, Input-state approach to Boolean networks, *IEEE Trans. Neural Networks*, vol. 20, no. 3, pp. 512-521, 2009. (**Regular Paper**)

Basic Control Properties

- Controllability under open-loop or closed-loop controls;
- Observability;
- Algebraic description of input-output transfer graph.

References:

D. Cheng, H. Qi, Controllability and observability of Boolean control networks, *Automatica*, vol. 45, no. 7, pp. 1659-1665, 2009. (**Regular Paper**)

Y. Zhao, H. Qi, D. Cheng, Input-state incidence matrix of Boolean control networks and its applications, *Sys. Contr. Lett.*, vol. 46, no. 12, pp. 767-774, 2010.

☞ System Realization

- State space expression;
- Input-output realization;
- Kalman decomposition, minimum realization.

References:

- D. Cheng, Z. Li, H. Qi, Realization of Boolean control networks, *Automatica*, vol. 46, no. 1, pp. 62-69, 2010. (**Regular Paper**)

- D. Cheng, H. Qi, State space analysis of Boolean network, *IEEE Trans. Neural Networks*, vol. 21, no. 4, pp. 584-594, 2010. (**Regular Paper**)

☞ Control Design

- Disturbance decoupling;
- Stability and stabilization;
- Canalizing mapping and its applications.

References:

D. Cheng, Disturbance Decoupling of Boolean control networks, *IEEE Trans. Aut. Contr.*, vol. 56, no. 1, pp. 2-10, 2011. (**Regular Paper**)

D. Cheng, H. Qi, Z. Li, J.B. Liu, Stability and stabilization of Boolean networks, *Int. J. Robust Nonlin. Contr.*, vol. 21, no. 2, pp. 134-156, 2001.

☞ Optimal Control

- Topological structure of Boolean control networks;
- Optimal control and its design.
- k- and Mix-valued and higher-order control networks.

References:

- Y. Zhao, Z. Li, D. Cheng, Optimal control of logical control networks *IEEE Trans. Aut. Contr.*, vol. 56, no. 8, pp. 1766-1776, (**Regular Paper**).
- Z. Li, D. Cheng, Algebraic approach to dynamics of multi-valued networks, *Int. J. Bifurcat. Chaos*, vol. 20, no. 3, pp. 561-582, 2010.

☞ Identification

- Identify the dynamic evolution;
- Identify via input-output data.

References:

D. Cheng, Y. Zhao, Identification of Boolean control networks, *Automatica*, vol. 47, no. 4, pp. 702-710, 2011.(**Regular Paper**)

D. Cheng, H. Qi, Z. Li, Model construction of Boolean network via observed data, *IEEE Trans. Neural Networks*, vol. 22, no. 4, pp. 525-536, 2011. (**Regular Paper**)

☞ My Book

Communications and Control Engineering
Daizhan Cheng
Hongsheng Qi
Zhiqiang Li
Analysis and
Control of Boolean
Networks
Semi-tensor Product Approach
Springer

IV. Dynamic Game

☞ Static Game

Definition 4.1

(1) A static game G consists of three ingredients: (i) n players, named $A_1, \cdots, A_n$; (ii) each player A_i has k_i possible actions, denoted by $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdot n$; (iii) n payoff functions for n players respectively as

$$c_j(x_1 = i_1, \cdots, x_n = i_n) = c^j_{i_1\ i_2\ \cdots\ i_n}, \quad j = 1, \cdots, n. \tag{31}$$

(2) A set of actions $s = (x_1, \cdots, x_n)$, is a strategy of G, denoted by S.

(3) A strategy $\{x_j^*\}$ is a Nash equilibrium if

$$c_j(x_1^*, \cdots, , x_j^*, \cdots, x_n^*) \geq c_j(x_1^*, \cdots, x_j, \cdots, x_n^*) \\ j = 1, \cdots, n. \tag{32}$$

Example 4.2

Prisoners' Dilemma

- Action 1: Confess
- Action 2: Deny

Table 4: Payoff bi-matrix

$P_1 \backslash P_2$	1	2
1	-3,-3	0,-5
2	-5,0	-1,-1

Nash Equilibrium is $(1, 1)$.

☞ Dynamic Game

$$G \Rightarrow G_\infty$$

Payoff Functions

$$J_j = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=1}^{T} c_j(x(t)), \quad j = 1, \cdots, n.$$

☞ Strategy with Finite Memory

Definition 4.3

A strategy for G

☞ Human-Machine Game

$$m(t+1) = f(m(t), h(t)), \tag{34}$$

$$J_h = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=1}^{T} c_h(x(t)).$$

Theorem 4.4

(1) The best strategy is state-control periodic.

(2) The best strategy ($h^*(t)$) satisfies

$$h^*(t+1) = g(m(t), h(t)) = Lm(t)h(t). \tag{35}$$

☞ Human-Machine Game (continued)

Find best strategy:

(1) find cycles on state-control space;

(2) find optimal L, where

$$L \in \mathcal{L}_{q \times pq},$$

where p: Number of machine strategies; q: Number of human strategies;

References:

Y. Zhao, Z. Li, D. Cheng, Optimal control of logical control networks, *IEEE Trans. Aut. Contr.*, vol. 56, no. 8, pp. 1766-1776, 2011 (**Regular Paper**).

☞ Mixed Strategy

Consider player i:

$$S_i = \{1, 2, \cdots, k_i\}$$

$$x_i = j, \quad \text{with Probability } p_i(j),$$

where $\sum_{j=1}^{k_i} p_i(j) = 1$.

- Finite Horizon case:

$$J_h = E\left[\sum_{t=1}^{N} \lambda^t c_h(h(t), m(t)) \Big| m(0)\right].$$

Here $0 < \lambda < 1$ (discount factor).

Theorem 4.5

Let $J^*(m(0))$ be the optimal value of J_h. Then

$$J^*(x(0)) = J_0(x(0)), \tag{36}$$

where the function J_0 is given by the last step of a dynamic programming algorithm. Setting $c_t := \lambda^t c(h(t), m(t))$, the algorithm proceeds backward in time from time step N to time step 0 as follows.

$$J_N(m(N)) = \max_{h(N)\in\Delta_r} c_t(h(N), m(N)). \tag{37}$$

and for $t = N-1, N-2, \cdots, 1, 0$:

$$J_t(m(t)) = \max_{h(t)\in\Delta_r} E\left[c_t(h(t), m(t)) + J_{t+1}(m(t+1)) \middle| m(t), h(t)\right]. \tag{38}$$

- Infinite Horizon case:

$$J_h = E\left[\sum_{t=1}^{\infty} \lambda^t c_h(h(t), m(t)) \middle| m(0)\right].$$

Receding Horizon Based Feedback Control:

Denote

$$\min_{h \in \Delta_k} \min_{h_i \neq h_j \in \Delta_r} \left| c(m, h_i) - c(m, h_j) \right| := d.$$

$$M := \max_{h \in \Delta_r, m \in \Delta_k} |c(h, m)| < \infty.$$

Theorem 4.6

Assume $d > 0$. Then the optimal control sequence $u^*(0), u^*(1), \cdots$ obtained by receding horizon control is exactly the optimal control for the infinite horizon case, provided that the prediction horizon length ℓ satisfies

$$\ell > \log_\lambda \frac{(1-\lambda)d}{2M}. \tag{39}$$

D. Cheng, Y. Zhao, T. Xu. Receding horizon based feedback optimization for mix-valued logical networks, *IEEE Trans. Aut. Contr.*, In press, On line: `http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=7079492`, DOI: 10.1109/TAC.2015.2419874.

☞ Networked Evolutionary Games

Definition 4.7

A networked evolutionary game (NEG), denoted by $\mathcal{G} = ((N,E), G, \Pi)$, consists of three factors:

(i) a network graph: (N,E);

(ii) a fundamental network game (FNG): G with two players. Players i and j play this game provided $(i,j) \in E$.

(iii) a local information based strategy updating rule (SUR):

$$x_i(t+1) = f_i(x_j(t), c_j(t) \mid j \in U(i)), \quad i = 1, \cdots, n. \tag{40}$$

D. Cheng, F. He, H. Qi, T. Xu. Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Aut. Contr.*, In press, On line: `http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=7042754`, DOI: 10.1109/TAC.2015.2404471. (**Regular Paper**)

V. Concluding Remarks

The algebraic state space representation of logical dynamic systems has various applications:

- (networked) evolutionary games;
- logical circuit design and related topics:
- cryptography:
- fuzzy control:
- graph theory and formation control:
- communication;
- control of power systems and engine transient control;

Thank you!

Question?