Potential Game and Its Application to Control

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Outline of Presentation

I. An Introduction to Game Theory

☞ Game Theory

Figure 1: John von Neumann

J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944.

☞ Non-Cooperative Game
(Winner of Nobel Prize in Economics 1994)

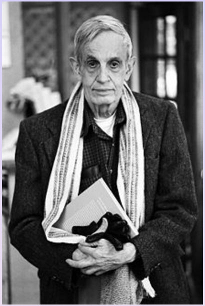


Figure 2: John Forbes Nash Jr.

J. Nash, Non-cooperative game, *The Annals of Mathematics*, Vol. 54, No. 2, 286-295, 1951.

☞ Cooperative Game
(Winner of Nobel Prize in Economics 2012 with Roth)

Figure 3: Lloyd S. Shapley

D. Gale, L.S. Shapley, Colle admissions and the stability of marriage, Vol. 69, American Math. Monthly, 9-15, 1962.

☞ Market Power and Regulation

(Winner of Nobel Prize in Economics 2014)

Figure 4: Jean Tirole

- D. Fudenberg and J. Tirole, Game Theory, MIT Press, Cambridge, MA, 1991.
- J. Tirole, The Theory of Industrial Organization, MIT Press, Cambridge, MA, 1988.

☞ Normal Non-cooperative Game

Definition 1.1

A **normal game** $G = (N, \mathcal{S}, c)$:

(i) Player: $N = \{1, 2, \cdots, n\}$.

(ii) Strategy:

$$\mathcal{S}_i = \mathcal{D}_{k_i}, \quad i = 1, \cdots, n,$$

where

$$\mathcal{D}_k := \{1, 2, \cdots, k\}.$$

(iii) Profile:

$$\mathcal{S} = \prod_{i=1}^{n} \mathcal{S}_i.$$

(iv) Payoff function:

$$c_j : \mathcal{S} \to \mathbb{R}, \quad j = 1, \cdots, n. \tag{1}$$

$$c := \{c_1, \cdots, c_n\}.$$

☞ Nash Equilibrium

Definition 1.2

In a normal game G, a profile

$$s = (x_1^*, \cdots, x_n^*) \in \mathcal{S}$$

is a **Nash equilibrium** if

$$c_j(x_1^*, \cdots, , x_j^*, \cdots, x_n^*) \geq c_j(x_1^*, \cdots, x_j, \cdots, x_n^*) \\ j = 1, \cdots, n. \tag{2}$$

☞ Nash Equilibrium

Example 1.3

Consider a game G with two players: P_1 and P_2:

- Strategies of P_1: $\mathcal{D}_2 = \{1, 2\}$;
- Strategies of P_2: $\mathcal{D}_3 = \{1, 2, 3\}$.

Table 1: Payoff bi-matrix

$P_1 \backslash P_2$	1	2	3
1	2, 1	3, 2	6, 1
2	1, 6	2, 3	5, 5

$(1, 2)$ is a Nash equilibrium.

☞ Mixed Strategies

Definition 1.4

Assume the set of strategies for Player i is

$$S_i = \{1, \cdots, k_i\}.$$

Then Player i may take $j \in S_i$ with probability $r_j \geq 0$, $j = 1, \cdots, k_i$, where

$$\sum_{j=1}^{k_i} r_j = 1.$$

Such a strategy is called a **mixed strategy**. Denote by

$$x_i = (r_1, r_2, \cdots, r_{k_i})^T \in \Delta(S_i).$$

Notations

- **Mixed Strategy:**

$$\Upsilon_k := \left\{ (r_1, r_2, \cdots, r_k)^T \mid r_i \geq 0, \ \sum_{i=1}^{k} r_i = 1 \right\}.$$

- **Probabilistic Matrix:**

$$\Upsilon_{m\times n} := \left\{ M \in \mathcal{M}_{m\times n} \;\middle|\; \mathrm{Col}(M) \subset \Upsilon_m \right\}.$$

- $$\mathbf{1}_m := (\underbrace{1, \cdots, 1}_{m})^T.$$

☞ Existence of Nash Equilibrium

Definition 1.5 (Nash 1950)

In the n-player normal game, $G = (N, S, c)$, if $|N|$ and $|S_i|$, $i = 1, \cdots, n$ are finite, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

II. Semi-tensor Product of Matrices

$$A_{m\times n} \times B_{p\times q} = ?$$

Definition 2.1

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \operatorname{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \qquad (3)$$

☞ Important Comments

1. When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
2. STP keeps almost all the major properties of the conventional matrix product available.
   - Associativity, Distributivity;
   - $(A \ltimes B)^T = B^T \ltimes A^T$;
   - $(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}; \cdots$.

☞ Logical Variable and Logical Matrix

- *Vector Form of Logical Variables*: $x \in \mathcal{D}_k = \{1, 2, \cdots, k\}$, we identify

$$i \sim \delta_k^i, \quad i = 1, \cdots, k,$$

where δ_k^i is the i th column of I_k. Then $x \in \Delta_k$, where $\Delta_k = \{\delta_k^1, \cdots, \delta_k^k\}$.

- *Logical Matrix*:

$$L = [\delta_m^{k_1}, \delta_m^{k_2}, \cdots, \delta_m^{k_n}],$$

shorthand form:

$$L = \delta_m[k_1, k_2, \cdots, k_n].$$

☞ Matrix Expression of Logical Functions

Theorem 2.1

Let $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$ be a set of logical variables.

- Let $f : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ and

$$y = f(x_1, \cdots, x_n). \tag{4}$$

Then there exists a unique matrix $M_f \in \mathcal{L}_{k_0 \times k}$ $(k = \prod_{i=1}^{n} k_i)$ such that in vector form

$$y = M_f \ltimes_{i=1}^{n} x_i := M_f x, \tag{5}$$

where $x = \ltimes_{i=1}^{n} x_i$. M_f is called the structure matrix of f, and (5) is the algebraic form of (4).

☞ Matrix Expression of Pseudo-logical Functions

Theorem 2.1(cont'd)

- Let $c : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathbb{R}$ and

$$h = c(x_1, \cdots, x_n). \tag{6}$$

Then there exists a unique (row) vector $V_c \in \mathbb{R}^k$, such that in vector form

$$h = V_c x, \tag{7}$$

V_c is called the structure vector of c, and (7) is the algebraic form of (6)

☞ Khatri-Rao Product

Definition 2.2

Let $A \in \mathcal{M}_{p\times m}$, $B \in \mathcal{M}_{q\times m}$. Then the Khatri-Rao product of A and B is defined as

$$M * N := [\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \cdots \mathrm{Col}_m(M) \ltimes \mathrm{Col}_m(N)]. \quad (8)$$

☞ Matrix Expression of Logical Mapping

Let $x_i, y_j \in \mathcal{D}_k$, $i = 1, \cdots, n$, $j = 1, \cdots, m$, and $F : \mathcal{D}_k^n \to \mathcal{D}_k^m$ be

$$y_j = f_j(x_1, \cdots, x_n), \quad j = 1, \cdots, m. \tag{9}$$

Then in vector form we have

$$y_j = M_j x, \quad j = 1, \cdots, m. \tag{10}$$

Theorem 2.3

F can be expressed as

$$y = M_F x. \tag{11}$$

where $y = \ltimes_{j=1}^m y_j$, and

$$M_F = M_1 * M_2 * \cdots * M_m \in \mathcal{L}_{2^m \times 2^n}. \tag{12}$$

III. Potential Games

☞ Vector Space Structure of Finite Games

- $\mathcal{G}_{[n;k_1,\cdots,k_n]}$: the set of finite games with $|N|=n$, $|S_i|=k_i$, $i=1,\cdots,n$;
- In vector form: $x_i \in S_i = \Delta_{k_i}$, $i=1,\cdots,n$;
- $c_i : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathbb{R}$ can be expressed (in vector form) as

$$c_i(x_1,\cdots,x_n) = V_i^c \ltimes_{j=1}^n x_j, \quad i=1,\cdots,n,$$

  where V_i^c is the structure vector of c_i.
- Set

$$V_G := [V_1^c, V_2^c, \cdots, V_n^c] \in \mathbb{R}^{nk}.$$

  Then each $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is uniquely determined by V_G. Hence, $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ has a natural vector structure as

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} \sim \mathbb{R}^{nk}.$$

☞ Potential Games

Definition 3.1

Consider a finite game $G = (N, S, C)$. G is a potential game if there exists a function $P : S \to \mathbb{R}$, called the potential function, such that for every $i \in N$ and for every $s^{-i} \in S^{-i}$ and $\forall x, y \in S_i$

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = P(x, s^{-i}) - P(y, s^{-i}), \quad i = 1, \cdots, n. \tag{13}$$

D. Monderer, L.S. Shapley, Potential Games, *Games and Economic Behavior*, Vol. 14, 124-143, 1996.

☞ Fundamental Properties

Theorem 3.2

If G is a potential game, then the potential function P is unique up to a constant number. Precisely if P_1 and P_2 are two potential functions, then $P_1 - P_2 = c_0 \in \mathbb{R}$.

Theorem 3.3

Every finite potential game possesses a pure Nash equilibrium. Certain evolutions (Sequential or cascading MBRA) lead to a Nash equilibrium.

D. Monderer, L.S. Shapley, Potential games, Games Econ. Theory, 97, 81-108, 1996.

☞ Is a Game Potential?

Numerical computation ($n = 2$):

- Shapley (96): $O(k^4)$;
- Hofbauer (02): $O(k^3)$;
- Hilo (11): $O(k^2)$;
- Cheng (14): Potential Equation.

Hilo: "It is not easy, however, to verify whether a given game is a potential game."

D. Monderer, L.S. Shapley, Potential games, Games Econ. Theory, 97, 81-108, 1996.

J. Hofbauer, G. Sorger, A differential game approach to evolutionary equilibrium selection, Int. Game Theory Rev. 4, 17-31, 2002.

Y. Hino, An improved algorithm for detecting potential games, Int. J. Game Theory, 40, 199-205, 2011.

D. Cheng, On finite potential games, *Automatica*, Vol. 50, No. 7, 1793-1801, 2014.

Lemma 3.4

G is a potential game if and only if there exist $d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n)$, which is independent of x_i, such that

$$\begin{aligned} c_i(x_1, \cdots, x_n) = & P(x_1, \cdots, x_n) \\ & + d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n), \quad i = 1, \cdots, n, \end{aligned} \tag{14}$$

where P is the potential function.

Structure Vector Express:

$$\begin{array}{lcl} c_i(x_1, \cdots, x_n) & := & V_i^c \ltimes_{j=1}^n x_j \\ d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) & := & V_i^d \ltimes_{j \neq i} x_j, \quad i = 1, \cdots, n, \\ P(x_1, \cdots, x_n) & := & V_P \ltimes_{j=1}^n x_j. \end{array}$$

Define:

$$k^{[p,q]} := \begin{cases} \prod_{j=p}^{q} k_j, & q \geq p \\ 1, & q < p. \end{cases}$$

Construct:

$$\begin{aligned} E_i &:= I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \\ &\in \mathcal{M}_{k \times k/k_i},\ i = 1, \cdots, n. \end{aligned} \tag{15}$$

Note that $\mathbf{1}_k \in \mathbb{R}^k$ is a column vector with all entries equal 1; $I_s \in \mathcal{M}_{s \times s}$ is the identity matrix and $I_1 := 1$.

$$\xi_i := \left(V_i^d\right)^T \in \mathbb{R}^{k^{n-1}}, \quad i = 1, \cdots, n. \tag{16}$$

☞ Potential Equation

Then (14) can be expressed as a linear system:

$$E\xi = b, \tag{17}$$

where

$$E = \begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}; \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}; \quad b = \begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix}. \tag{18}$$

(17) is called the potential equation and Ψ is called the potential matrix.

☞ Main Result

Theorem 3.5

A finite game G is potential if and only if the potential equation has solution. Moreover, the potential P can be calculated by

$$V_P = V_1^c - V_1^d (E_1)^T = V_1^c - \xi_1^T \left(\mathbf{1}_k^T \otimes I_k \right). \tag{19}$$

Example 3.6

Consider a prisoner's dilemma with the payoff bi-matrix as in Table 2.

Table 2: Payoff Bi-matrix of Prisoner's Dilemma

$P_1 \backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

Example 3.6 (cont'd)

From Table 2

$$V_1^c = (R, S, T, P)$$
$$V_2^c = (R, T, S, P).$$

Assume $V_1^d = (a, b)$ and $V_2^d = (c, d)$. It is easy to calculate that

$$E_1 = \delta_2[1, 2, 1, 2]^T,$$
$$E_2 = \delta_2[1, 1, 2, 2]^T.$$

$$b_2 = (V_2^c - V_1^c)^T = (0, T - S, S - T, 0)^T.$$

Example 3.6 (cont'd)

Then the potential equation (18) becomes

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ T-S \\ S-T \\ 0 \end{bmatrix}. \tag{20}$$

Example 3.6 (cont'd)

It is easy to solve it out as

$$\begin{cases} a = c = T - c_0 \\ b = d = S - c_0 \end{cases}$$

where $c_0 \in \mathbb{R}$ is an arbitrary number. We conclude that the general **Prisoner's Dilemma is a potential game**.
Using (19), the potential can be obtained as

$$\begin{aligned} V_P &= V_1^c - V_1^d D_f^{[2,2]} \\ &= (R - T, 0, 0, P - S) + c_0(1, 1, 1, 1). \end{aligned} \quad (21)$$

From (17), G is potential if and only if

$$\begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \in \mathrm{Span}(E). \tag{22}$$

Since V_1^c is free, we have

$$\begin{bmatrix} (V_1^c)^T \\ (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \in \mathrm{Span}(E^e), \tag{23}$$

where

$$E^e = \begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}.$$

Equivalently, we have

$$\begin{bmatrix} I_k & 0 & \cdots & 0 \\ -I_k & I_k & \cdots & 0 \\ \vdots & & \ddots & \\ -I_k & 0 & \cdots & I_k \end{bmatrix} \begin{bmatrix} (V_1^c)^T \\ (V_2^c)^T \\ (V_3^c)^T \\ \vdots \\ (V_n^c)^T \end{bmatrix} \in \mathrm{Span}(E^e). \tag{24}$$

That is

$$V_G^T \in \mathrm{Span}(E_P), \tag{25}$$

where

$$
\begin{aligned}
E_P &:= \begin{bmatrix} I_k & 0 & \cdots & 0 \\ -I_k & I_k & \cdots & 0 \\ \vdots & & \ddots & \\ -I_k & 0 & \cdots & I_k \end{bmatrix}^{-1} E^e \\
&= \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}.
\end{aligned}
\tag{26}
$$

E_n^0 is obtained from E_n by deleting the last column, and define

$$E_P^0 := \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n^0 \end{bmatrix}.$$

Then we have

$$\mathrm{Span}(E_P) = \mathrm{Span}(E_P^0).$$

Moreover, it is easy to see that the columns of E_P^0 are linearly independent.

☞ Potential Subspace

Theorem 3.7

The subspace of potential games is

$$\mathcal{G}_P = \mathrm{Span}(E_P), \tag{27}$$

which has $\mathrm{Col}(E_P^0)$ as its basis.

According to the construction of E_P^0 it is clear that

Corollary 3.8

The dimension of the subspace of potential games of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ is

$$\dim\left(\mathcal{G}_P\right) = k + \sum_{j=1}^{n} \frac{k}{k_j} - 1. \tag{28}$$

IV. Decomposition of Finite Games

☞ Non-strategic Games

Definition 4.1

Let $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. G and $\tilde{G}$ are said to be strategically equivalent, if for any $i \in N$, any $x_i,\ y_i \in S_i$, and any $x^{-i} \in S^{-i}$, (where $S^{-i} = \prod_{j \neq i} S_j$), we have

$$c_i(x_i, x^{-i}) - c_i(y_i, x^{-i}) = \tilde{c}_i(x_i, x^{-i}) - \tilde{c}_i(y_i, x^{-i}), \quad i = 1, \cdots, n. \tag{29}$$

Lemma 4.2

Two games $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ are strategically equivalent, if and only if for each $x^{-i} \in S^{-i}$ there exists $d_i(x^{-i})$ such that

$$\begin{aligned} & c_i(x_i, x^{-i}) - \tilde{c}_i(x_i, x^{-i}) = d_i(x^{-i}), \\ & \forall x_i \in S_i,\ \forall x^{-i} \in S^{-i},\ i = 1, \cdots, n. \end{aligned} \tag{30}$$

Theorem 4.3

G and $\tilde{G}$ are strategically equivalent if and only if

$$\left(V_G^c - V_{\tilde{G}}^c\right)^T \in \mathrm{Span}\left(B_N\right), \tag{31}$$

where

$$B_N = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{32}$$

Definition 4.4

The subspace

$$\mathcal{N} := \mathrm{Span}(B_N)$$

is called the non-strategic subspace.

Corollary 4.5

The dimension of $\mathcal{N}$ is

$$\dim(\mathcal{N}) = \sum_{i=1}^{n} \frac{k}{k_i}. \quad (33)$$

Define

$$\tilde{E}_P := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{34}$$

Comparing (34) with (26), it is ready to verify that

$$\mathcal{G}_P = \mathrm{Span}\left(\tilde{E}_P\right) = \mathrm{Span}\left(E_P\right). \tag{35}$$

Deleting the last column of $\tilde{E}_P$, (equivalently, replacing the E_n in $\tilde{E}_P$ by E_n^0), the remaining matrix is denoted as

$$\tilde{E}_P^0 := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n^0 \end{bmatrix}. \qquad (36)$$

Then it is clear that $\mathrm{Col}\left(\tilde{E}_P^0\right)$ is a basis of $\mathcal{G}_P$.

Observing (34) again, it follows immediately that

Corollary 4.6

The subspace $\mathcal{N}$ is a linear subspace of $\mathcal{G}_P$. That is,

$$\mathcal{N} \subset \mathcal{G}_P.$$

☞ Orthogonal Decomposition

Theorem 4.7

(Candogan et al, 2011)

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{Potential\quad games}\overbrace{\oplus \quad \mathcal{H}}^{Harmonic\quad games}. \tag{37}$$

O. Candogan, I. Menache, A. Ozdaglar, P.A. Parrilo, Flows and decompositions of games: Harmonic and potential games, *Mathematcs of Operations Research*, Vol. 36, No. 3, 474-503, 2011.

☞ Pure Potential Games $\mathcal{P}$

Using (34)-(35), we have

$$
\begin{aligned}
\mathcal{G}_P &= \mathrm{Span}(\tilde{E}_P) \\
&= \mathrm{Span}\begin{bmatrix}
I_k - \frac{1}{k_1}E_1E_1^T & E_1 & 0 & 0 & \cdots & 0 \\
I_k - \frac{1}{k_2}E_2E_2^T & 0 & E_2 & 0 & \cdots & 0 \\
I_k - \frac{1}{k_3}E_3E_3^T & 0 & 0 & E_3 & \cdots & 0 \\
\vdots & & & & \ddots & \\
I_k - \frac{1}{k_n}E_nE_n^T & 0 & 0 & 0 & \cdots & E_n
\end{bmatrix}.
\end{aligned} \tag{38}
$$

$$B_P = \begin{bmatrix} I_k - \frac{1}{k_1} E_1 E_1^T \\ I_k - \frac{1}{k_2} E_2 E_2^T \\ \vdots \\ I_k - \frac{1}{k_n} E_n E_n^T \end{bmatrix} \in \mathcal{M}_{nk \times k}. \tag{39}$$

Then we have

$$\mathcal{P} = \mathcal{V} = \mathrm{Span}\left(B_P\right). \tag{40}$$

Since $\dim(\mathcal{P}) = k-1$, to find the basis of $\mathcal{P}$ one column of V needs to be removed. Note that

$$
\begin{aligned}
& \left(I_k - \tfrac{1}{k_i} E_i E_i^T\right) \mathbf{1}_k \\
= \; & \left(I_{k[1,i-1]} \mathbf{1}_{k[1,i-1]}\right) \left[\left(I_{k_i} - \tfrac{1}{k_i} \mathbf{1}_{k_i \times k_i}\right) \mathbf{1}_{k_i}\right] \\
& \left(I_{k[i+1,n]} \mathbf{1}_{k[i+1,n]}\right) \\
= \; & 0, \quad i = 1, \cdots, n.
\end{aligned}
$$

It follows that

$$B_P \mathbf{1}_{nk} = 0.$$

Deleting any one column of B_P, say, the last column, and denoting the remaining matrix by B_P^0, then we know that

Theorem 4.8

$$\mathcal{P} = \mathrm{Span}\left(B_P\right) = \mathrm{Span}\left(B_P^0\right),$$

where B_P^0 is a basis of $\mathcal{P}$.

☞ Pure Harmonic Games $\mathcal{H}$

we can construct a set of vectors, which are in $\mathcal{G}_P^{\perp}$ as

$$J_1 := \left\{ \begin{array}{c} \begin{bmatrix} (\delta^1_{k_1} - \delta^{i_1}_{k_1})(\delta^1_{k_2} - \delta^{i_2}_{k_2})\delta^{i_3}_{k_3} \cdots \delta^{i_n}_{k_n} \\ -(\delta^1_{k_1} - \delta^{i_1}_{k_1})(\delta^1_{k_2} - \delta^{i_2}_{k_2})\delta^{i_3}_{k_3} \cdots \delta^{i_n}_{k_n} \\ \mathbf{0}_{(n-2)k} \end{bmatrix} \\ i_1 \neq 1, i_2 \neq 1 \end{array} \right\};$$

$$J_2 := \left\{ \begin{array}{c} \begin{bmatrix} (\delta^1_{k_1} - \delta^{i_1}_{k_1})\delta^1_{k_2}(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^{i_4}_{k_4}\cdots\delta^{i_n}_{k_n} \\ \delta^{i_1}_{k_1}(\delta^1_{k_2} - \delta^{i_2}_{k_2})(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^{i_4}_{k_4}\cdots\delta^{i_n}_{k_n} \\ -(\delta^1_{k_1}\delta^1_{k_2} - \delta^{i_1}_{k_1}\delta^{i_2}_{k_2})(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^{i_4}_{k_4}\cdots\delta^{i_n}_{k_n} \\ \mathbf{0}_{(n-3)k} \end{bmatrix} \\ (i_1, i_2) \neq \mathbf{1}_2^T; i_3 \neq 1 \end{array} \right\};$$

$$\vdots$$

$$
J_{n-1} :=
\left\{ \begin{bmatrix}
(\delta^1_{k_1} - \delta^{i_1}_{k_1})\delta^1_{k_2}\delta^1_{k_3}\delta^1_{k_4}\cdots\delta^1_{k_{n-1}}(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
\delta^{i_1}_{k_1}(\delta^1_{k_2} - \delta^{i_2}_{k_2})\delta^1_{k_3}\delta^1_{k_4}\cdots\delta^1_{k_{n-1}}(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^1_{k_4}\cdots\delta^1_{k_{n-1}}(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
\vdots \\
\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\delta^{i_3}_{k_3}\delta^{i_4}_{k_4}\cdots(\delta^1_{k_{n-1}} - \delta^{i_{n-1}}_{k_{n-1}})(\delta^1_{k_n} - \delta^{i_n}_{k_n}) \\
-(\delta^1_{k_1}\delta^1_{k_2}\cdots\delta^1_{k_{n-1}} - \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\cdots\delta^{i_{n-1}}_{k_{n-1}})(\delta^1_{k_n} - \delta^{i_n}_{k_n})
\end{bmatrix} \\
(i_1, \cdots, i_{n-1}) \neq \mathbf{1}^T_{n-1}; i_n \neq 1 \right\}.
$$

Define

$$B_H := [J_1, J_2, \cdots, J_{n-1}]. \tag{41}$$

Then we can show B_H is the basis of $\mathcal{H}$:

Theorem 4.9

B_H has full column rank and

$$\mathcal{H} = \mathrm{Span}(B_H). \tag{42}$$

Theorem 4.10

$G \in \mathcal{H}$, iff

- $$\sum_{i=1}^{n} c_i(s) = 0, \quad s \in S; \tag{43}$$

- $$\sum_{x \in S_i} c_i(x, y) = 0, \quad \forall y \in S^{-i}; \ i = 1, \cdots, n. \tag{44}$$

☞ Nash Equilibrium of $\mathcal{G}_H$

Definition 4.11

Let $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ and $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ a Nash equilibrium of G. s^* is called a flat Nash equilibrium, if

$$\begin{aligned} c_i(s_1^*, s_2^*, \cdots, s_n^*) &= c_i(s_1^*, s_2^*, \cdots, s_i, \cdots, s_n^*), \\ & \forall s_i \in S_i;\ i = 1, \cdots, n. \end{aligned}$$

A flat Nash equilibrium is called a zero Nash equilibrium if

$$c_i(s_1^*, s_2^*, \cdots, s_n^*) = 0, \quad i = 1, \cdots, n.$$

Example 4.12

Consider $G \in \mathcal{G}_{[2;k_1,k_2]}$. Assume (s_1^*, s_2^*) is a flat Nash equilibrium, then the payoff bi-matrix is as Table 3:

Table 3: Flat Nash Equilibrium

$P_1 \backslash P_2$	1	2	$\cdots$	s_2^*	$\cdots$	k_2
1	$(\times, \times)$	$(\times, \times)$	$\cdots$	$(a, \times)$	$\cdots$	$(\times, \times)$
2	$(\times, \times)$	$(\times, \times)$	$\cdots$	$(a, \times)$	$\cdots$	$(\times, \times)$
$\vdots$			$\vdots$		$\vdots$	
s_1^*	$(\times, b)$	$(\times, b)$	$\cdots$	(a, b)	$\cdots$	$(\times, b)$
$\vdots$			$\vdots$		$\vdots$	
k_1	$(\times, \times)$	$(\times, \times)$	$\cdots$	$(a, \times)$	$\cdots$	$(\times, \times)$

As $a = b = 0$, (s_1^*, s_2^*) is a zero Nash equilibrium.

☞ Nash Equilibriums of $\mathcal{G}_H = \mathcal{H} \oplus \mathcal{N}$

Theorem 4.13

1. If $G \in \mathcal{N}$, then every strategy profile is a flat Nash equilibrium;
2. If $G \in \mathcal{H}$ and s^* is a Nash equilibrium, then s^* is a zero Nash equilibrium;
3. If $G \in \mathcal{G}_H$ and s^* is a Nash equilibrium, then s^* is a flat Nash equilibrium.

☞ Networked Evolutionary Game (NEG)

Definition 5.1

A **networked evolutionary game**, denoted by $((N,E),G,\Pi)$, consists of

(i) a network graph (N,E);

(ii) a fundamental network game (FNG), G, such that if $(i,j) \in E$, then i and j play FNG with strategies $x_i(t)$ and $x_j(t)$ respectively;

(iii) a local information based strategy updating rule (SUR).

☞ Network Graph: (N, E)

Definition 5.2

1. (N, E) is a graph, where N is the set of nodes and $E \subset N \times N$ is the set of edges.
2. $U_d(i) = \{j \,|\,$ there is a path connecting i, j with length $\leq d\}$
3. $U_0(i) := \{i\}; \quad U_1(i) = U(i); \quad U_\alpha(i) \subset U_\beta(i),\ \alpha \leq \beta.$
4. If $(i,j) \in E$ implies $(j,i) \in E$ the graph is undirected, otherwise, it is directed.

Definition 5.3

A network is **homogeneous**, if each node has the same degree (for undirected graph) / in-degree and out-degree (for directed graph).

☞ Fundamental Network Game: G

Definition 5.4

A normal game with two players is called a **fundamental network game** (FNG), if

$$S_1 = S_2 := S_0 = \{1, 2, \cdots, k\}.$$

☞ Overall Payoff

$$c_i(t) = \sum_{j \in U(i) \backslash i} c_{ij}(t), \quad i \in N. \tag{45}$$

☞ Strategy Updating Rule: Π

Definition 5.5

A **strategy updating rule** (SUR) for an NEG, denoted by Π, is a set of mappings:

$$x_i(t+1) = g_i\left(x_j(t), c_j(t) \,\middle|\, j \in U(i)\right), \quad t \geq 0, \quad i \in N. \tag{46}$$

Remark 5.6

1. g_i could be a probabilistic mapping (*i.e.*, a mixed strategy is used);
2. When the network is homogeneous, g_i, $i \in N$, are the same.

☞ Strategy Profile Dynamics

Since $c_j(t)$ depends on $x_\ell(t)$, $\ell \in U(j)$, (46) can be expressed as

$$x_i(t+1) = f_i\left(x_j(t) \,\middle|\, j \in U_2(i)\right), \quad t \geq 0, \quad i \in N. \tag{47}$$

Now (47) is a standard k-valued logical dynamic system, its profile dynamics can be expressed as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)). \end{cases} \tag{48}$$

D. Cheng, F. He, H. Qi, T. Xu. Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Aut. Contr.*, (in print), On line: DOI:10.1109/TAC.2015.2404471.

☞ Potential NEG

Theorem 5.7

Consider an NEG, $((N,E),G,\Pi)$. If the fundamental network game G is potential, then the NEG is also potential. Moreover, the potential P of the NEG is:

$$P(s) := \sum_{(i,j)\in E} P^{i,j}(s_i, s_j). \tag{49}$$

Example 5.8

Consider an NEG $((N, E), G, \Pi)$, where the network graph is described as in Fig. 5.

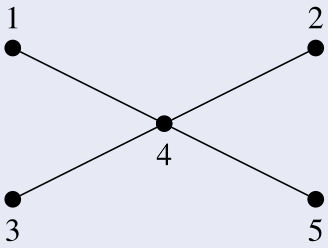


Figure 5: Network Graph

Example 5.8 (cont'd)

Assume:

- G: the prisoner's dilemma with $R = -1$, $S = -10$, $T = 0$, $P = -5$.
- Π: MBRA (Potential $\Rightarrow$ Pure Nash Equalibrium)

$$\Psi = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{128\times 80}.$$

Example 5.8 (cont'd)

It is easy to check that

$$V_1^c = \begin{bmatrix} -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & -5 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & 5 \end{bmatrix}.$$

$$V_2^c = \begin{bmatrix} -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & -1 & -1 & -10 & -10 \\ 0 & 0 & -5 & -5 & 0 & 0 & -5 & -5 \end{bmatrix}.$$

Example 5.8 (cont'd)

$$V_3^c = \begin{bmatrix} -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \\ -1 & -1 & -10 & -10 & 0 & 0 & -5 & -5 \end{bmatrix}.$$

$$V_4^c = \begin{bmatrix} -4 & -13 & 0 & -5 & -13 & -22 & -5 & -10 \\ -13 & -22 & -5 & -10 & -22 & -31 & -10 & -15 \\ -13 & -22 & -5 & -10 & -22 & -31 & -10 & -15 \\ -22 & -31 & -10 & -15 & -31 & -40 & -15 & -20 \end{bmatrix}.$$

Example 5.8 (cont'd)

$$V_5^c = [-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5$$
$$-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5$$
$$-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5$$
$$-1 \quad 0 \quad -10 \quad -5 \quad -1 \quad 0 \quad -10 \quad -5].$$

It is easy check that the networked game is potential.

Example 5.8 (cont'd)

Moreover,

$$\xi_1 = \begin{bmatrix} 28 & 27 & 15 & 10 & 27 & 26 & 10 & 5 \\ 27 & 26 & 10 & 5 & 26 & 25 & 5 & 0 \end{bmatrix}.$$

Using potential formula, we have

$$V_P = \begin{bmatrix} -29 & -28 & -25 & -20 & -28 & -27 & -20 & -15 \\ -28 & -27 & -20 & -15 & -27 & -26 & -15 & -10 \\ -28 & -27 & -20 & -15 & -27 & -26 & -15 & -10 \\ -27 & -26 & -15 & -10 & -26 & -25 & -10 & -5 \end{bmatrix}.$$

Example 5.8 (cont'd)

Calculating P separately.
First, for any $(i,j) \in E$ we have

$$P(x_i, x_j) = V_0 x_i x_j, \tag{50}$$

where

$$V_0 = (R-T, 0, 0, P-S) = (-1\ 0\ 0\ 5).$$

Next, we have

$$\begin{aligned} V_P^{1,2} &= V_0 D_r^{[4,8]} = V_0 \left(I_4 \otimes \mathbf{1}_8^T\right) \\ &= [-1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \\ &\quad\ \ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ &\quad\ \ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ &\quad\ \ 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5]. \end{aligned}$$

Example 5.8 (cont'd)

Similarly, we can figure out all $V_P^{i,j}$ as

$$
\begin{aligned}
V_P^{1,3} &= V_0 D_r^{[2,2]} D_r^{[8,2]}, & V_P^{1,4} &= V_0 D_r^{[2,4]} D_r^{[16,2]}, \\
V_P^{1,5} &= V_0 D_r^{[2,8]}, & V_P^{2,3} &= V_0 D_f^{[2,2]} D_r^{[8,4]}, \\
V_P^{2,4} &= V_0 D_f^{[2,2]} D_r^{[4,2]} D_r^{[16,2]}, & V_P^{2,5} &= V_0 D_f^{[2,2]} D_r^{[4,4]} \\
V_P^{3,4} &= V_0 D_f^{[4,2]} D_r^{[16,2]}, & V_P^{3,5} &= V_0 D_f^{[4,2]} D_r^{[8,2]}, \\
V_P^{4,5} &= V_0 D_f^{[8,2]}.
\end{aligned}
$$

Example 5.8 (cont'd)

$$\begin{aligned} V_{\tilde{P}} &= V_P^{1,4} + V_P^{2,4} + V_P^{3,4} + V_P^{4,5} \\ &= [\begin{matrix} -4 & -3 & 0 & 5 & -3 & -2 & 5 & 10 \\ -3 & -2 & 5 & 10 & -2 & -1 & 10 & 15 \\ -3 & -2 & 5 & 10 & -2 & -1 & 10 & 15 \\ -2 & -1 & 10 & 15 & -1 & 0 & 15 & 20 \end{matrix}]. \end{aligned}$$

Comparing this result with the above V_P, one sees easily that

$$\tilde{P}(x) = P(x) + 25.$$

VI. Applications

☞ Consensus of MAS

- Network graph: $(N, E(t))$: $N = \{1, 2, \cdots, n\}$ with varying topology: $E(t)$.
- Model of MAS:

$$a_i(t+1) = f_i\left(a_j(t) | j \in U(i)\right), \quad i = 1, \cdots, n. \tag{51}$$

- Set of Strategies:

$$a_i \in \mathcal{A}_i \subset \mathbb{R}^n, \quad i = 1, \cdots, n.$$

J.R. Marden, G. Arslan, J. S. Shamma, Cooperative control and potential games, *IEEE Trans. Sys., Man, Cybernetcs, Part B*, Vol. 39, No. 6, 1393-1407, 2009.

☞ Distributed Coverage of Graphs

- Unknown connected graph $\mathcal{G} = (V, E)$.
- Mobile agents $N = \{1, 2, \cdots, n\}$ (initially arbitrarily deployed on $\mathcal{G}$).
- Agent a_i can cover $U^i(t) := U_{d_i}(a_i(t))$, $i = 1, \cdots, n$.

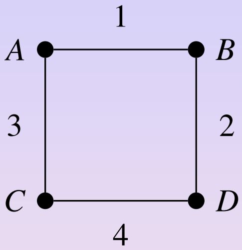
Purpose: $\max_a \bigcup_{i=1}^n U^i$.

A.Y. Yazicioglu, M. Egerstedt, J.S. Shamma, A game theoretic approach to distributed coverage of graphs by heterogeneous mobile agents, *Est. Contr. Netw. Sys.*, Vol. 4, 309-315, 2013.

M. Zhu, S. Martinez, Distributed coverage games for energy-aware mobile sensor networks, *SIAM J. Cont. Opt.*, Vol. 51, No. 1, 1-27, 2013.

☞ Congestion Games

Problem: Player 1 want to go from A to D, player 2 want to go from B to C:



- D. Monderer, L.S. Shapley, Potential Games, *Games & Economic Behavior*, Vol. 14, 124-143, 1996.
- X. Wang, N. Xiao, et al, Distributed consensus in noncooperative congestion games: an application to road pricing, *Proc. 10th IEEE Int. Conf. Contr. Aut.*, Hangzhou, China, 1668-1673, 2013.

V. Conclusion

- Formulas for verifying and calculating potential function are obtained.
- Vector space structure of finite non-cooperative games is introduced. Its decomposition is investigated.

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{Potential \quad games}\!\!\!\!\!\!\!\!\!\!\overbrace{\phantom{\mathcal{N}} \quad \oplus \quad \mathcal{H}}^{Harmonic \quad games}.$$

- The Nash equilibriums of $\mathcal{G}_H = \mathcal{H} \oplus \mathcal{N}$ are explored.
- The strategy profile dynamics of an NEG is derived. Properties of certain (potential) NEGs are studied.
- Three applications for potential NEGs are introduced.

Last Comments:

Game-based Control or Control Oriented Game could be a challenging new direction for Control Community.

Thank you for your attention!

Question?